Multiplier of periodic orbits in spaces of rational maps

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Dedicated to the memory of Adrien Douady

Abstract

Given a polynomial or a rational function f we include it in a space of maps. We introduce local coordinates in this space, which are essentially the set of critical values of the map. Then we consider an arbitrary periodic orbit of f with multiplier $\rho \neq 1$ as a function of the local coordinates, and establish a simple connection between the dynamical plane of f and the function ρ in the space associated to f. The proof is based on the theory of quasiconformal deformations of rational maps. As a corollary, we show that multipliers of non-repelling periodic orbits are also local coordinates in the space.

1 Introduction

The multiplier map of an attracting periodic orbit of a quadratic polynomial within the quadratic family $z \mapsto z^2 + v$ uniformizes the component of parameters v, for which it is attracting. This theorem [4], [29], [3] is a cornerstone of the Douady-Hubbard theory of the Mandelbrot set. It has been generalized to components of hyperbolic polynomials [24] and degree two rational maps [31].

In this paper we develop a local approach to the problem of behavior of multipliers of periodic orbits in general spaces of polynomials and rational maps. Our approach is somewhat closer to [9] and especially [13]. As a corollary, we show that the multiplier maps of attracting and neutral periodic orbits are local coordinates in the (moduli) spaces of polynomials and rational maps. It includes in particular the cited above Douady-Hubbard-Sullivan theorem.

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Let us state our main results. Let f be a rational function of degree $d \geq 2$. We consider a *Ruelle transfer operator* $T = T_f$, which acts on functions ψ as follows:

$$T\psi(z) = \sum_{w: f(w)=z} \frac{\psi(w)}{(f'(w))^2},$$

provided z is not a critical value of f. Let us also consider any *periodic orbit* $O = \{b_k\}_{k=1}^n$ of f of exact period n. Denote its *multiplier* by ρ:

$$\rho = (f^n)'(b_k) = \Pi_{j=1}^n f'(b_j).$$

We assume that $\rho \neq 1$. (For $\rho = 1$, see Sects. 2.3, 5.4.) Let us associate to the periodic orbit O of f a rational function B_O:

$$B_O(z) = \sum_{k=1}^n \frac{\rho}{(z-b_k)^2} + \frac{1}{1-\rho} \sum_{k=1}^n \frac{(f^n)''(b_k)}{z-b_k}. \quad (1)$$

It was introduced in [13] for the unicritical family.

On the other hand, we include f in a natural space of rational maps of the same degree. Roughly speaking, this is the set of maps with fixed multiplicities at different critical points and similar behavior at ∞. For instance, the space associated to a unicritical polynomial $z^d + v_0$ is the unicritical family $z^d + v$, $v \in \mathbf{C}$. We introduce local coordinates in the space near f, which are basically the set of critical values $v_1, ..., v_p$. When f, hence, the periodic orbit, moves in the space, ρ becomes a holomorphic function in these coordinates. Our main aim is to show that the following connection holds between the dynamical and the parameter spaces of f:

$$B_O(z) - (TB_O)(z) = \sum_{j=1}^p \frac{\partial \rho}{\partial v_j} \frac{1}{z - v_j}. \quad (2)$$

For unicritical polynomials, this connection appeared and was proved in [13]. It has been applied to the problems of geometry of Julia and the Mandelbrot sets in [13], [14], [15].

Comment 1 *Considered on quadratic differentials $\psi(z)dz^2$ the operator T is a so-called pushforward operator $\sum_{f(w)=z} \psi(w)dw^2$ introduced to the field probably by Thurston in his work on critically finite branched covering maps of the sphere, see [6]. In completely different applications it appeared in [12], [11], [35]. Explicit formulae for the Fredholm determinant of T in spaces of analytic functions have been proved and used in [20], [19], [18], [7]. See also [36]. The action of T on quadratic differentials with multiple poles was first studied and used in [9]. For applications to the problems of rigidity in complex dynamics, see [6], [33], [9], [26],*

[17], [27], [28], [37], [14], [15]. In particular, in [17] we use the operator T for proving the absence of invariant linefield on some Julia sets. The scheme of [17] is applied in [27], [28], [37].

Comment 2 *The function B_O associated to the periodic orbit O appears naturally in [13] in the context of quadratic polynomials $f(z) = z^2 + v$. Namely, assume that the periodic orbit O of period n is attracting, more exactly, $0 < |\rho| < 1$. Consider a series $H_\lambda(z) = \sum_{k>0} \frac{\lambda^{k-1}}{(f^{k-1})'(v)(z - f^k(0))}$, which converges for $|\lambda| < |\rho|^{1/n}$. Then the following identity holds ([12], [11], see also [35] [20]): $\lambda(TH_\lambda)(z) = H_\lambda(z) - \frac{D(\lambda)}{z-v}$, where $D(\lambda) = \sum_{k>0} \lambda^{k-1}/(f^{k-1})'(v)$. For a fixed z, the functions $H_\lambda(z)$ and $D(\lambda)$ extend to meromorphic functions on the complex plane, with simple poles at the points λ, so that $\lambda^n = \rho^{1-j}$, for $j = 0, 1, 2, \ldots$. Then calculations show that the residue of $H_\lambda(z)$ at the point $\lambda = 1$ is (up to a factor) the rational function $B_O(z)$. Taking the residues at $\lambda = 1$ of both sides of the above identity, we come to the relation $TB_O(z) = B_O(z) - \frac{L}{z-v}$, for some number L. Furthermore, it is shown in [13] that the latter holds for any periodic orbit of f with $\rho \neq 1$, and that $L = \rho'(v)$. Surprisingly, all this is generalized by the connection (2) of the present paper to a non-linear polynomial and rational function, with special normalization at infinity. The proof also sheds light on the nature of (2).*

The idea of the proof of (2) is roughly as follows, see Sects 3.2, 6 for details, for polynomials and rational functions respectively. Given f along with its periodic orbit O, we join it by a path inside of the space associated to f to a map g, such that g is hyperbolic, and the analytic continuation of O along the path turns O into an attracting periodic orbit of g. Since both sides of (2) depend analytically on the local coordinates, it follows that it is enough to prove (2) for an open subset of hyperbolic maps g and for their attracting periodic orbits O. To do this, we use the theory of quasiconformal deformations ("Teichmuller theory") of rational maps developed in Mane-Sad-Sullivan [22] and McMullen-Sullivan [29]. The operator T will serve as a transfer between parameters and the dynamics in this context.

We show also that multipliers of non-repelling periodic orbits are local coordinates in the space associated to f modulo a standard equivalence relation. For precise statements, see Theorem 2 for polynomials, and Sect. 5.5 for rational functions. We illustrate the results on the examples of quadratic polynomials (Comments 7-8) and quadratic rational functions (Corollary 5.1). The proof of Theorems 2 and 6 is based on (2), see Sects. 4 and 11 respectively. Then it boils down to the fact that T has no fixed points of a certain form (which is roughly a linear combination of functions B_O for non-repelling orbits), and this follows from the contraction property of T. The latter idea goes back to Thurston's work mentioned above and has been applied, among others, in [6], [33], [36], [9], [17], [13], [27]. See end of Sect. 2.2, Sect. 4 and particularly Sect. 11 for precise formulations and further discussion.

In Sects. 2-4 we accomplish these aims for polynomials, see Theorems 1, 2, and in Sects. 5-11 we do this for rational maps, see Theorem 5 and Sect. 5.5. Although the proof for polynomials and rational maps is essentially the same, we treat the polynomial case separately because of a special characteristic behavior of polynomials at ∞, and also because the proof in this case is technically simpler and hence more transparent.

Comment 3 *Since (2) is a formal identity, which holds for any rational function over* $\mathbf{C}$ *and any periodic orbit with multiplier not equal to one, it holds (literally) for rational functions over every field which is isomorphic to* $\mathbf{C}$*, in particular, for the p-adic fields.*

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Throughout the paper,

$$B(a,r) = \{z : |z-a| < r\}, \quad B^*(R) = \{z : |z| > R\}.$$

2 Polynomials. Formulation of main results

2.1 Polynomial spaces

Introduce a space $\Pi_{d,\bar{p}}$ of polynomials and its subspace $\Pi^q_{d,\bar{p}}$ as follows. Roughly speaking, the first space is the set of maps with fixed multiplicities of critical points, and maps from its subspace are those with a fixed number of different critical values.

Definition 2.1 *Let* $d \geq 2$ *be an integer,* $\bar{p}$ *a set of* p *positive integers* $\bar{p} = \{m_j\}_{j=1}^p$*, such that* $\sum_{j=1}^p m_j = d-1$*, and* q *an integer* $1 \leq q \leq p$*. A polynomial* f *of degree* d *belongs to* $\Pi_{d,\bar{p}}$ *iff it is monic and centered, i.e., has the form*

$$f(z) = z^d + a_1 z^{d-2} + ... + a_{d-1},$$

and, moreover, f *has* p *geometrically different critical points* $c_1, ..., c_p$ *with the multiplicities (as roots of* f'*)* $m_1, ..., m_p$ *resp.*

The space $\Pi^q_{d,\bar{p}}$ is said to be a subset of those $f \in \Pi_{d,\bar{p}}$, such that f has precisely q geometrically different critical values, i.e., the set $\{v_j = f(c_j),\ j = 1, ..., p\}$ contains q different points.

In particular, $m_j = 1$ iff c_j is simple, so that $p = d - 1$ iff all critical points are simple. At the other extreme case, the space $\Pi_{d,\bar{1}}$ consists of the unicritical family $z^d + v$. Up to a linear change of variables, every non-linear polynomial belongs to some $\Pi^q_{d,\bar{p}}$.

By definition,

$$f'(z) = d.\Pi_{j=1}^p (z - c_j)^{m_j}. \tag{3}$$

Since f is centered, there is a linear connection between $c_1, ..., c_p$:

$$\sum_{j=1}^p m_j c_j = 0. \tag{4}$$

Let us identify $f \in \Pi_{d,\bar{p}}$ as above with the point

$$\bar{f} = \{a_1, ..., a_{d-1}\} \in \mathbf{C}^{d-1}.$$

Given $f_0 \in \Pi_{d,\bar{p}}$, we introduce two local coordinates $\bar{c}$ and $\bar{v}$ in a neighborhood of f_0 in $\Pi_{d,\bar{p}}$. Roughly speaking, $\bar{c}$ encodes a map through geometrically different critical points, and $\bar{v}$ through their images (corresponding critical values). Let us define it precisely.

Let $\{c_1(f_0), ..., c_p(f_0)\}$ be the collection of all geometrically different critical points of f_0. We fix the order of the critical points (so called "marked polynomial"). Then, for every $f \in \Pi_{d,\bar{p}}$, such that $\bar{f}$ and $\bar{f}_0$ are close enough points of $\mathbf{C}^p$, the critical points $c_1(f), ..., c_p(f)$ of f can be ordered in such a way, that $c_j(f)$ is close to $c_j(f_0)$, $j = 1, ..., p$. We set

$$\bar{c}(f) = \{f(0), c_1, ..., c_p\}, \tag{5}$$

where $c_j = c_j(f)$. Note that $c_1, ..., c_p$ satisfy (4). It follows from the equality $f(z) = f(0) + d\int_0^z \Pi_{j=1}^p (w - c_j)^{m_j} dw$, that f is defined uniquely by $\bar{c}$ and, moreover, $\bar{f} \in \mathbf{C}^{d-1}$ is a holomorphic function of $\bar{c}$ modulo the linear relation (4). Clearly, $\bar{c}$ lies in the p-dimensional linear subspace of $\mathbf{C}^{p+1}$, and so we identify this subspace with $\mathbf{C}^p$.

We are going to introduce a second coordinate system $\bar{v}$ in a neighborhood of f_0, and prove that it is indeed a local coordinate. In fact, $\bar{v}$ will play a central role for us. In the previous notations, we define

$$\bar{v} = \bar{v}(f) = \{v_1, ..., v_p\}, \quad v_j = \bar{v}_j(f) = f(c_j(f)). \tag{6}$$

Let us stress that some critical values v_j might coincide, so that the number $q(f)$ of geometrically different critical values of f might be less than p. Nevertheless, the point $\bar{v}$ is a point of $\mathbf{C}^p$.

We have a well-defined correspondence $\pi : \bar{c} \mapsto \bar{v}$ from a neighborhood of the point $\bar{c}(f_0)$ in $\mathbf{C}^p$ into a neighborhood of $v(f_0) = \pi(\bar{c}(f_0))$ in $\mathbf{C}^p$.

Proposition 1 *1. The map*

$$\pi : \bar{c} \mapsto \bar{v}$$

is a local biholomorphic homeomorphism of a neighborhood of $\bar{c}(f_0)$ in $\mathbf{C}^p$ onto a neighborhood of $\bar{v}(f_0)$.

2. $\bar{f}$ is a local holomorphic function of $\bar{v}$. It is locally biholomorphic if all the critical points of f_0 are simple.

We prove it in Sect. 3.1.

The space $\Pi_{d,\bar{p}}$ can therefore be identified in a neighborhood of its point f_0 with a neighborhood $W_{f_0}(\epsilon) = \{(v_1, ..., v_p) : |v_j - v_j(f_0)| < \epsilon\}$, $\epsilon > 0$, in $\mathbf{C}^p$. If, moreover, $f_0 \in \Pi^q_{d,\bar{p}}$, for some q, then its neighborhood in $\Pi^q_{d,\bar{p}}$ is the intersection of $W_{f_0}(\epsilon)$ with a q-dimensional linear subspace of $\mathbf{C}^p$ defined by the conditions: $v_i = v_j$ iff $v_i(f_0) = v_j(f_0)$. Thus the vector $\{V_1, ..., V_q\}$ of different critical values of $f \in \Pi^q_{d,\bar{p}}$ serves as a local coordinate systems in $\Pi^q_{d,\bar{p}}$.

2.2 Main results

Let f be a polynomial. Consider a periodic orbit $O = \{b_k\}_{k=1}^n$ of f of exact period n. Denote its multiplier by ρ:

$$\rho = (f^n)'(b_k) = \Pi_{j=1}^n f'(b_j).$$

We assume that $\rho \neq 1$. (For $\rho = 1$, see next Subsect. 2.3.) Suppose f is monic and centered. Then it belongs to some space $\Pi_{d,\bar{p}}$ and, moreover, to its subspace $\Pi^q_{d,\bar{p}}$. (In fact, these spaces are defined uniquely, up to the order of the different critical points.) These will be the parameter spaces associated to f. By the Implicit Function theorem and by Proposition 1, there is a set of n functions $O(\overline{v}) = \{b_k(\overline{v})\}_{k=1}^n$ defined and holomorphic in $\overline{v} \in \mathbf{C}^\mathbf{p}$ in a neighborhood of $\bar{v}(f)$, such that $O(\overline{v}) = O$ for $\overline{v} = \overline{v}(f)$, and $O(\bar{v})$ is a periodic orbit of $g \in \Pi_{d,\bar{p}}$ of period n, where g is in a neighborhood of f, and $\overline{v} = \overline{v}(g)$. In particular, if $\rho(\overline{v})$ denotes the multiplier of the periodic orbit $O(\overline{g})$ of g, it is a holomorphic function of $\overline{v}$ in this neighborhood. The standard notation $\partial\rho/\partial v_j$ denotes the partial derivatives.

Now, suppose g stays in a neighborhood of f inside of $\Pi^q_{d,\bar{p}}$. Then the multiplier ρ of $O(g)$ is, in fact, a holomorphic function of the vector of q different critical values $\{V_1, ..., V_q\}$ of g. By $\partial^V\rho/\partial V_k$ we then denote the partial derivatives of ρ w.r.t. these critical values. We have:

$$\frac{\partial^V \rho}{\partial V_k} = \sum_{j:v_j=V_k} \frac{\partial \rho}{\partial v_j}. \tag{7}$$

The operator T associated to f and the rational function $B = B_O$ associated to O are defined in the Introduction.

Theorem 1 *Suppose $f \in \Pi_{d,\bar{p}}$, so that $c_1, ..., c_p$ denote all geometrically different critical points of f, and $v_j = f(c_j)$ corresponding critical values (not necessarily different). Let O be a periodic orbit of f with multiplier $\rho \neq 1$, and $B = B_O$. Then*

$$B(z) - (TB)(z) = \sum_{j=1}^{p} \frac{\partial \rho}{\partial v_j} \frac{1}{z - v_j}, \tag{8}$$

and

$$\frac{\partial \rho}{\partial v_j} = -\frac{1}{(m_j - 1)!} \frac{d^{m_j - 1}}{dw^{m_j - 1}}|_{w=c_j} \frac{B(w)}{Q_j(w)} = -\frac{1}{2\pi i} \int_{|w-c_j|=r} \frac{B(w)}{f'(w)} dw, \tag{9}$$

where c_j is a critical point of f of multiplicity m_j, and Q_j is a polynomial defined by $f'(z) = (z - c_j)^{m_j} Q_j(z)$. Furthermore, if $f \in \Pi^q_{d,\bar{p}}$, then

$$B(z) - (TB)(z) = \sum_{k=1}^{q} \frac{\partial^V \rho}{\partial V_k} \frac{1}{z - V_k}, \tag{10}$$

where V_k, $k = 1, ..., q$, are all pairwise different critical values of f.

In view of (7), the formula (10) is an immediate corollary of (8).

For the family of unicritical polynomials, i.e., in the space $\Pi_{d,\bar{1}}$, the formula (8) appears for the first time in [13]. Note that its proof in [13] is formal, and in this sense "mysterious", as it is pointed out there. Our proof resolves in some way this "mystery". It is based on the Teichmuller theory of rational maps [29], [22]. The bridge between this theory and our problem is provided by the following property of the operator T from the space $L_1(\mathbf{C})$ into itself: the adjoint operator of T is an operator T^* in $L_\infty(\mathbf{C})$ acting as follows: $T^*\nu = |f'|^2/f'^2 \nu \circ f$. A fixed point ν of T^* is called an *invariant Beltrami form* of f, see [22], [29]. To be more precise, we will make use of the following. A backward invariant Beltrami form on a set V is a function $\mu \in L_\infty(V)$, for which $\mu(f(x))|f'(x)|^2/(f'(x))^2 = \mu(x)$, for a.e. $x \in f^{-1}(V)$. For every function ψ, which is integrable on V, we then have (by change of variable):

$$\int_V \mu(z) T\psi(z) d\sigma_z = \int_{f^{-1}(V)} \mu(z)\psi(z) d\sigma_z. \tag{11}$$

Here and below $d\sigma_z$ denotes the area element on the z-plane. We have similarly that T is a contraction in a sense that

$$\int_V |T\psi| d\sigma \leq \int_{f^{-1}(V)} |\psi| d\sigma. \tag{12}$$

2.3 Cusps

Here we consider the remaining case $\rho = 1$, under the assumption that the periodic orbit is *non-degenerate*. In other words, we assume that the periodic orbit $O = \{b_1, ..., b_n\}$ of f of the exact period n has the multiplier $\rho = 1$, and $(f^n)''(b_j) \neq 0$, for some (hence, for any) $j = 1, ..., n$. Then, for any polynomial g, which is close to f, the map g in a small neighborhood of O has either precisely two different periodic orbits $O_g^{\pm}$ of period n with multipliers $\rho^{\pm} \neq 1$, or precisely one periodic orbit O_g of period n with the multiplier 1.

Now, assume that $f \in \Pi^q_{d,\bar{p}}$, and let f_i, $i = 1, 2, ...$, be any sequence of maps from $\Pi_{d,\bar{p}}$, such that $f_i \to f$, $i \to \infty$. We assume that each f_i has a periodic orbit O_i near O, such that its multiplier $\rho_i \neq 1$. In other words, the orbit O_i is either $O^+_{f_i}$ or $O^-_{f_i}$. Introduce

$$\hat{B}_i(z) = (1 - \rho_i) B_{O_i}(z) = \sum_{b \in O_i} \{\frac{\rho_i(1-\rho_i)}{(z-b)^2} + \frac{(f_i^n)''(b)}{z-b}\}. \tag{13}$$

As $i \to \infty$, we have obviously that $\hat{B}_i$ tend to a rational function $\hat{B} = \hat{B}_O$, where

$$\hat{B}(z) = \sum_{b \in O} \frac{(f^n)''(b)}{z-b}.$$

Now, multiplying both hand-sides of (8) for f_i and B_{O_i} by $1 - \rho_i$ and passing to the limit as $i \to \infty$, we get:

Proposition 2 *For every $j = 1, ..., p$, the following finite limit exists:*

$$\frac{\hat{\partial}\rho}{\partial v_j} := \lim_{i \to \infty} (1 - \rho_i) \frac{\partial \rho_i}{\partial v_j}. \tag{14}$$

Then we have:

$$\hat{B}(z) - (T\hat{B})(z) = \sum_{j=1}^{p} \frac{\hat{\partial}\rho}{\partial v_j} \frac{1}{z - v_j}. \tag{15}$$

The formula (9) holds, where one replaces ρ and B by $\hat{\rho}$ and $\hat{B}$ respectively. Furthermore, if f and f_i are in $\Pi^q_{d,\bar{p}}$, then, for every $j = 1, ..., p$, there exists a finite limit

$$\frac{\hat{\partial}^V \rho}{\partial v_j} := \lim_{i \to \infty} (1 - \rho_i) \frac{\partial^V \rho_i}{\partial v_j}, \tag{16}$$

and

$$\hat{B}(z) - (T\hat{B})(z) = \sum_{k=1}^{q} \frac{\hat{\partial}^V \rho}{\partial V_k} \frac{1}{z - V_k}. \tag{17}$$

Comment 4 *We will see (take $r = 1$ in the next Theorem 2) that the vectors $\{\hat{\partial}\rho/\partial v_j\}_{j=1}^p$ and $\{\hat{\partial}^V \rho/\partial V_j\}_{j=1}^q$ are non-zero.*

2.4 Multipliers and local coordinates

Theorem 1, Proposition 2 and the contraction property of T yield the following.

Theorem 2 *Suppose that $f \in \Pi^q_{d,\bar{p}}$ has a collection $O_1, ..., O_r$ of r different periodic orbits with the corresponding multipliers $\rho_1, ..., \rho_r$, such that each O_j is non-repelling: $|\rho_j| \le 1$, $j = 1, ..., r$. Assume that, if, for some j, $\rho_j = 1$, then the periodic orbit O_j is non-degenerate. Denote by $\tilde{\partial}^V \rho_j / \partial V_k$ the $\partial^V \rho_j / \partial V_k$ iff $\rho_j \ne 1$ and $\hat{\partial}^V \rho_j / \partial V_k$ iff $\rho_j = 1$. With these notations, introduce the following matrix $\mathbf{O}$:*

$$\mathbf{O} = (\frac{\tilde{\partial}^V \rho_j}{\partial V_1}, ..., \frac{\tilde{\partial}^V \rho_j}{\partial V_q})_{1 \le j \le r}. \tag{18}$$

Assume furthermore that, for every $j = 1, ..., r$, either $\rho_j \ne 0$, or, if $\rho = 0$, then the periodic orbit O_j contains a single critical point, and this critical point is simple. Then the rank of the matrix $\mathbf{O}$ is equal to r, that is, maximal.

Comment 5 *By the Fatou-Douady-Shishikura inequality, see [3], [34], $r \le q$, the number of geometrically different critical values of the polynomial f. See also Comment 9 as well as Comments 10 and 11.*

Comment 6 *Assume in Theorem 2 that $\rho_j \ne 1$ for $j = 1, ..., r$. Applying in this case the Implicit Function theorem to the matrix $\mathbf{O}$ of rank r, we obtain that one can define a new local coordinate system in $\Pi^q_{d,\bar{p}}$ by replacing r coordinates in $\bar{V} = (V_1, ...V_q)$ by r multipliers of different non-repelling periodic orbits with the multipliers not equal to 1. In fact, the case when some $\rho_j = 1$ can also be included replacing ρ_j by $-\{(1-\rho_j^+)^2 + (1-\rho_j^-)^2\}/4$ in a neighborhood of f, where $\rho_j^\pm$ are the multipliers of the periodic orbits $O_j^\pm$ of nearby maps defined in the previous Sect. 2.3.*

Comment 7 *Theorem 2 contains as a particular case the Douady-Hubbard-Sullivan Theorem: the multiplier map of an attracting periodic orbit of the map $z^2 + v$ is an isomorphism of the corresponding hyperbolic component of the Mandelbrot set onto the unit disk [4], [3], [29]. For other generalizations of this important result for polynomials, see [23]. Note that the case $f \in \Pi^q_{d,\bar{p}}$ and $r = q$, under the assumption that the periodic orbits $O_1, ..., O_r$ are attracting follows also from a general result on hyperbolic polynomials proved in [23]. Note however that the method of quasiconformal surgery used in [4], [3], [29], [23] breaks down in the presence of a neutral periodic orbit. Our result is completely general. On the other hand, it is local.*

Comment 8 *Consider another particular case: $f_{v_0}(z) = z^d + v_0$ and O_0 is a non-degenerate periodic orbit of f_{v_0} with the multiplier $\rho_0 = 1$. Then the matrix $\mathbf{O}$ is one-dimensional and consists of the number*

$$\hat{\rho}'(v_0) := \lim_{v \to v_0} (1 - \rho(v)) \rho'(v), \tag{19}$$

where $\rho(v)$ is the multiplier of a periodic orbit of $f_v(z) = z^d + v$, $v \neq v_0$, which is close to O_0. By Theorem 2, $\hat{\rho}'(v_0) \neq 0$, which means that the corresponding hyperbolic component of the connectedness locus of the family f_v has a cusp at the point v_0. The latter is proven in [5] (for $d = 2$) using global considerations.

3 Theorem 1

3.1 Proof of Proposition 1

The map π is locally well-defined and holomorphic, because the coefficients of f are holomorphic functions of $\bar{c}$. It maps a neighborhood of $\bar{c}$ in $\mathbf{C}^p$ into $\mathbf{C}^p$. Therefore, to prove that π is locally biholomorphic, it is enough to show that π is a local injection (see e.g. [38], Chapter 4, Theorem 1V). On the other hand, the latter follows essentially from the Monodromy Theorem. Here is a detailed proof.

Fix $f_0 \in \Pi_{d,\bar{p}}$. It has p geometrically different critical points $c_j(f_0)$ and $q_0 = q(f_0)$ geometrically different critical values $v_1^0, ..., v_{q_0}^0$, $q_0 \leq p$. Choose a covering of the Riemann sphere $\hat{\mathbf{C}}$ by a finite collection of (open) balls $B_1, ..., B_m$ centered at some points $a_1, ..., a_m$, as follows.

(1) For $1 \leq k \leq q_0$, the ball B_k is centered at the critical value v_k^0 of f_0, i.e., $a_k = v_k^0$, and the closures $\bar{B}_k, \bar{B}_i$, for $1 \leq i < k \leq q_0$, are pairwise disjoint.

(2) B_m is centered at infinity: $a_m = \infty$. For every $1 \leq k \leq m$, the center a_k of B_k is away from the closure $\bar{B}_i$ of any other ball B_i, $k \neq i$.

(3) For every $1 \leq k \leq m$, and every component U of $f_0^{-1}(B_k)$, the following holds: either U is disjoint from the set of critical points of f_0, or U contains one and only one critical point $c_j(f_0)$ of f_0. In the former case, the map $f_0 : U \to B_k$ is univalent, and in the latter case, $f_0 : U \to B_k$ is an $m_j + 1$-branched covering, with a single critical point at $c_j(f_0)$.

By this, every component U of $f_0^{-1}(B_k)$ contains one and only one preimage w_U of a_k by f_0. Call w_U the "center" of the component U. Denote by $d(z, w)$ the spherical distance between z, w in the Riemann sphere. Consider any $f \in \Pi_{d,\bar{p}}$, such that $\bar{f}$ and $\bar{f}_0$ are close points in $\mathbf{C}^{d-1}$. By definition, f has p geometrically different critical points $c_j(f)$ with the corresponding multiplicities m_j, and $c_j(f)$ is close to $c_j(f_0)$, $1 \leq j \leq p$. Note however, that the number $q(f)$ of geometrically different critical values of f can be larger than the number q_0 of geometrically different critical values of f_0. We have, by the above, similar properties for preimages of B_k by f:

(1f) For every $1 \leq k \leq m$, and every component $U(f)$ of $f^{-1}(B_k)$ the following holds. The set $U(f)$ contains one and only one "center" w_U of some component U of $f_0^{-1}(B_k)$. Moreover, $U(f)$ and U are close (in, say, Hausdorff distance).

(2f) There are two possibilities: (a) if $w_U \in U(f)$ is not a critical point of f_0, then the map $f : U(f) \to B_k$ is univalent, (b) if $w_U = c_j(f_0)$, for some j, then

$f: U \to B_k$ is an m_j+1-branched covering, with the single critical point at $c_j(f)$.

To prove the injectivity, consider two maps f_1, f_2 in $\Pi_{d,\bar{p}}$, so that $\bar{c}(f_1)$, $\bar{c}(f_2)$ are close to $\bar{c}(f_0)$, and assume that

$$\bar{v}(f_1) = \bar{v}(f_2). \tag{20}$$

We know that $\bar{f}$ is a continuous (even holomorphic) function of $\bar{c}$. Hence, $\bar{f}_1$, $\bar{f}_2$ are close to $\bar{f}_0$, too. We get from (1f)-(2f):

(3f) fix $r > 0$ small enough (smaller than half of the spherical distance between any point w_U and any component of $f_0^{-1}(B_k)$ other than U). For every $1 \le k \le m$, there exists $\epsilon_k > 0$ with the following property. For a component U of $f_0^{-1}(B_k)$, if $\bar{f}_1$, $\bar{f}_2$ are close enough to $\bar{f}_0$, and if there is a point $\hat{z} \in U$, such that $d(\hat{z}, w_U) > r$ and $d(F \circ f_1(\hat{z}), \hat{z}) < \epsilon_k/2$, for some branch F of f_2^{-1} in B_k, then:

(i) $F \circ f_1$ is a (well-defined) holomorphic function in the component $U(f_1)$ which contains w_U,

(ii) $d(F \circ f_1(z), z) < \epsilon_k$, for all $z \in U(f_1)$.

Now, let $\bar{f}_i$, $i = 1, 2$ be close enough to $\bar{f}_0$. Fix ϵ_* positive and less than $\epsilon_k/2^{md}$, for all k. Let us start with a branch $F_\infty(z) = z^{1/d} + ...$ of f_2^{-1} in B_m, such that $g = F_\infty \circ f_1$ is well-defined near infinity, and $g(z) = z + O(1/z)$ at ∞. Then g extends to a holomorphic function in $U_\infty(f_1)$. Since $\bar{f}_i$, $i = 1, 2$, and $\bar{f}_0$ are close enough, then $d(g(\hat{z}), \hat{z}) < \epsilon_*$, for any $\hat{z}$ in the intersection of $U_\infty(f_1)$ and any component $V(f_1)$ of any other $f_1^{-1}(B_k)$. By (3f), g extends to a holomorphic function along every chain of components of $f_1^{-1}(B_k)$ that form a connected set, therefore, g is holomorphic in $\mathbf{C}$. By the normalization, it is the identity map, which proves that $f_1 = f_2$.

The second part is obvious because if the (finite) critical points of f are simple, then $\bar{f}$ is a local biholomorphic map of $\bar{c}$. By the first part, we are done.

3.2 Reductions

By the continuity of functions ρ and $\partial\rho/\partial v_j$ in $\bar{v}$, it is enough to prove the formulae of Theorem 1 assuming that $\rho \neq 0$.

The identity. Denote

$$A(z) = \frac{1}{\rho} B(z) = \sum_{k=1}^{n} \frac{1}{(z - b_k)^2} + \frac{1}{\rho(1-\rho)} \sum_{k=1}^{n} \frac{(f^n)''(b_k)}{z - b_k}. \tag{21}$$

First, we will prove the following general identity about the rational functions fixing infinity.

Theorem 3 *Let f be any rational function so that ∞ is a fixed point (possibly, superattracting) of f. Let c_j, $j = 1, ..., p$ be all geometrically different finite critical*

points of f, and assume that the corresponding critical values $v_j = f(c_j)$, $j = 1, ..., p$, are also finite. Then there are numbers $L_1, ..., L_p$, such that

$$A(z) - (TA)(z) = \sum_{j=1}^{p} \frac{L_j}{z - v_j}. \tag{22}$$

For $j = 1, ..., p$, the coefficient

$$L_j = -\frac{1}{(m_j - 1)!} \frac{d^{m_j - 1}}{dw^{m_j - 1}}|_{w=c_j}(\frac{A(w)}{Q_j(w)}), \tag{23}$$

where Q_j is a local analytic function near c_j defined by $f'(z) = (z - c_j)^{m_j} Q_j(z)$, so that $Q_j(c_j) \neq 0$. In particular, if c_j is simple, then $L_j = -A(c_j)/f''(c_j)$.

Reduction to the hyperbolic case. Here we show that it is enough to prove Theorem 1 only for those f from $\Pi_{d,\bar{p}}$ that satisfy the following conditions:

(1) f is a hyperbolic map, moreover, O is an attracting periodic orbit of f, which attracts all critical points c_j, $j = 1, ..., p$,

(2) f has no critical relations between critical points except for the constant multiplicities of the critical points themselves: $f^n(c_i) = f^m(c_j)$ if and only if $i = j$ and $m = n$.

Indeed, assume that Theorem 1 holds for this subset of maps from $\Pi_{d,\bar{p}}$. Note that it is open in $\Pi_{d,\bar{p}}$. Given now any f as in Theorem 1, we find a real analytic simple path $\gamma : [0, 1] \to \Pi_{d,\bar{p}}$, $\gamma(t) = g_t$, which obeys the following properties: (i) $g_0 = f$, (ii) g_1 satisfies conditions (1)-(2), (iii) the analytic continuation O_t (a periodic orbit of g_t) of the periodic orbit O along the path is well-defined (i.e., the multiplier of O_t is not 1 for $t \in [0, 1]$), and O_1, a periodic orbit of g_1, is attracting. All critical points of g_1 are attracted to O_1.

Let us for a moment take for granted the existence of such path. Since the critical points of $g_t = \gamma(t)$ change continuously along γ and don't collide, the coordinate system $\bar{v}$ can be chosen changing continuously in a whole neighborhood of γ. Fix z. Denote by $\Delta(z, \overline{v})$ the difference between the left and the right hand sides of (8). It is an analytic function in $\bar{v}$ in a neighborhood of every point $\bar{v}(g_t)$, $t \in [0, 1]$. On the other hand, by the assumption that the theorem holds for maps satisfying (1)-(2), it is identically zero in a neighborhood of $\bar{v}(g_1)$. By the Uniqueness Theorem for analytic functions, $\Delta(z, \bar{v}) = 0$.

Let us show that the path γ as above does exist. First, we need the following fact about the parameter space of the unicritical family $p_c(z) = z^d + c$. Consider any repelling periodic orbit Q of the map $p_0(z) = z^d$. There is a real analytic simple path $p_{c_Q(t)}$, $c_Q : [0, 1] \to \mathbf{C}$, such that there exists an analytic continuation of Q to a periodic orbit Q_t of $p_{c_Q(t)}$, $0 \leq t \leq 1$, such that $Q_0 = Q$ and Q_1 is attractive. Indeed, if we assume the contrary that such a path does not exist, then

the Monodromy Theorem ensures the existence of an analytic continuation of Q to the whole complex plane. Then the multiplier of this continuation is an entire function in c that omits values in the unit disk, which is impossible. Thus the path $p_{c_Q(t)}$ as above does exist.

Note also that it is easy to find maps from $\Pi_{d,\bar{p}}$ near every p_c.

Let us come back to the construction of the path γ. Firstly, we connect f to the map p_0 by a path γ_0 from $[0,1]$ to the space of polynomials of degree d, so that $\gamma_0(t) \in \Pi_{d,\bar{p}}$ for $0 \le t < 1$. We construct γ_0 explicitly as follows. For $\tau \in \mathbf{C}$, set $c_j(\tau) = (1-\tau)c_j(f)$, $1 \le j \le p$. For every τ, define a polynomial

$$F_\tau(z) = (1-\tau)f(0) + d. \int_0^z \Pi_{j=1}^p (w - c_j(\tau))^{m_j} dw. \tag{24}$$

Note that $F_\tau \in \Pi_{d,\bar{p}}$ for every $\tau \neq 0$. Then a real analytic simple curve $\tau : [0,1] \to \mathbf{C}$, $\tau(0) = 0$, $\tau(1) = 1$, can be chosen so that the analytic continuation O_t of the periodic orbit O of f along the path $\gamma_0(t) = F_{\tau(t)}$ exists, and O_1 is some periodic orbit Q of p_0. We proceed by a real analytic path c_Q in the parameter plane of p_c that turns Q into an attracting periodic orbit of some p_c. Finally, we find the desired path γ in $\Pi_{d,\bar{p}}$ in a neighborhood of $c_Q \circ \gamma_0$.

Hyperbolic maps Here we describe how to prove Theorem 1 for the hyperbolic maps f. We assume that $f \in \Pi_{d,\bar{p}}$ *satisfies the conditions (1)-(2) of the previous paragraph.* To clarify the meaning of the coefficients L_j in (22), we will use the theory of quasiconformal deformations of rational maps [29]. The main technical part is contained in the next Theorem 4. By a *Beltrami coefficient* we mean a measurable function $\nu(z)$ on the Riemann sphere, such that $|\nu(z)| \le k < 1$ for almost every z. Let $\nu(z,t)$ be an *analytic family of invariant Beltrami coefficients.* By this we mean a family $\nu_t(z) = \nu(z,t)$ of Beltrami coefficients on the Riemann sphere, which is analytic in t as a map from a neighborhood of $t = 0$ into $L_\infty(\mathbf{C})$, and such that, for every t, ν_t is an invariant Beltrami form of f. We always assume $\nu(z,0) = 0$. Assume, additionally, $\nu(z,t) = 0$ for z in the basin of infinity of f. Let h_t be an analytic in t family of quasiconformal homeomorphism in the plane tangent to ∞, so that h_t has the complex dilatation $\nu(z,t)$ (i.e., $\nu(t,z) = \frac{\partial h_t}{\partial \bar{z}} / \frac{\partial h_t}{\partial z}$), and $h_0 = id$. Set $f_t = h_t \circ f \circ h_t^{-1}$. It is well-known that then $\{f_t\}$ is an analytic family of polynomials. Since f_t is conjugated to f, then $f_t \in \Pi_{d,\bar{p}}$. Let $O_t = h_t(O)$ be the corresponding attracting periodic orbit of f_t. Denote by $\rho(t)$ its multiplier, and by $v_j(t) = h_t(v_j)$ the critical values of f_t. Based on Theorem 3, we derive:

Theorem 4

$$\frac{\rho'(0)}{\rho} = \sum_{j=1}^p L_j v_j'(0). \tag{25}$$

Concluding argument. Here we show how Theorem 4 implies Theorem 1. We are going to compare (25) to the following obvious identity:

$$\rho'(0) = \sum_{j=1}^{p} \frac{\partial \rho}{\partial v_j} v_j'(0). \tag{26}$$

The proof will be finished once we will show that the vector $\{v_1'(0), ..., v_p'(0)\}$ for $f \in \Pi_{d,\bar{p}}$ can be chosen arbitrary.

For every vector $\bar{v}' = \{v_1', ..., v_p'\} \in \mathbf{C}^p$ of initial conditions, there exists an analytic family f_t of polynomials from $\Pi_{d,\bar{p}}$ with different critical points $c_1(t), ..., c_p(t)$ and corresponding critical values $v_1(t), ..., v_p(t)$, $v_j(t) = f_t(c_j(t)$, such that $v_j'(0) = v_j'$, for $1 \le j \le p$. Indeed, this is an immediate consequence of Proposition 1, where one can simply take locally $\bar{v}(t) = \bar{v} + t\bar{v}'$, and find by the inverse holomorphic correspondence $\bar{v} \mapsto \bar{c}$ a local family f_t, such that $f_0 = f$. Recall that $f \in \Pi_{d,\bar{p}}$ is a hyperbolic polynomial, which has no critical relations except for constant multiplicities at the critical points and such that no critical point of f is attracted by ∞. Then so is the conjugated map f_t, for every t close to 0. In particular, the basin of infinity of f_t is simply-connected on the Riemann sphere. We construct a holomorphic motion h_t of the plane as follows. First, for every t, define h_t in the basin of infinity of f to be $h_t = B_{f_t}^{-1} \circ B_f$, where B_P denotes the Bottcher coordinate function of a polynomial P such that $B_P(z)/z \to 1$ as $z \to \infty$. Note that h_t is holomorphic in the basin of ∞. Then we define h_t on the grand orbits of the critical points of f as in the proof of Theorem 7.4 of [29]. By Theorem 3 of [2], the holomorphic motion h_t extends in a unique way to a holomorphic motion of the plane, which we again denote by h_t, such that the complex dilatation of h_t is harmonic. As it is shown in [29], this h_t agrees with the dynamics. By Theorem 3 of [2], the complex dilatation $\nu(t, z)$ of h_t depends holomorphically on t. It vanishes in the basin of infinity of f because h_t is holomorphic there.

This proves the existence of $\nu(z, t)$ as above, which determines $v_j(t)$ with prescribed values $v_j'(0) = v_j'$, $j = 1, ..., p$. By this we finish the proof of the implication that Theorem 4 implies Theorem 1.

3.3 Proof of Theorem 3

Action of T on Cauchy kernels.

Lemma 3.1 *Let f be as in Theorem 3, and $a \in \mathbf{C}$ a parameter. Assume all finite critical points c_j, $j = 1, ..., p$, are simple, and the corresponding critical values $v_j = f(c_j)$ are finite. Then*

$$T\frac{1}{z-a} = \frac{1}{f'(a)} \frac{1}{z - f(a)} + \sum_{j=1}^{N} \frac{1}{f"(c_j)(c_j - a)} \frac{1}{z - v_j}. \tag{27}$$

Moreover,

$$T\frac{1}{(z-a)^2} = \frac{1}{(z-f(a))^2} - \frac{f"(a)}{(f'(a))^2}\frac{1}{z-f(a)} + \sum_{j=1}^{N}\frac{1}{f"(c_j)(c_j-a)^2}\frac{1}{z-v_j}. \quad (28)$$

Proof. Consider the integral

$$I = \frac{1}{2\pi i}\int_{|w|=R}\frac{dw}{f'(w)(f(w)-z)(w-a)}$$

and apply the Residue Theorem. It gives (27). Taking the derivative of (27) with respect to the parameter a, we get (28).

$\square$

Proof in the case of simple critical points. Recall that

$$A(z) = \sum_{k=1}^{n}\frac{1}{(z-b_k)^2} + \sum_{k=1}^{n}\frac{\gamma_k}{z-b_k},$$

where $O = \{b_1, ..., b_n\}$ is a periodic orbit of f of exact period n and with the multiplier $\rho \neq 1, 0$, and

$$\gamma_k = \frac{(f^n)"(b_k)}{\rho(1-\rho)}.$$

Assuming all critical points are simple, we can apply Lemma 3.1, and see that

$$(TA)(z) = \sum_{k=1}^{n}\frac{1}{(z-f(b_k))^2} + \sum_{j=1}^{n}\frac{\frac{\gamma_k}{f'(b_k)} - \frac{f"(b_k)}{(f'(b_k))^2}}{z-f(b_k)} + \sum_{j=1}^{d-1}\frac{A(c_j)}{f"(c_j)}\frac{1}{z-v_j}. \quad (29)$$

Therefore,

$$A(z) - (TA)(z) = \sum_{k=1}^{n}\frac{\gamma_{k+1} - \frac{\gamma_k}{f'(b_k)} + \frac{f"(b_k)}{(f'(b_k))^2}}{z-b_{k+1}} - \sum_{j=1}^{d-1}\frac{A(c_j)}{f"(c_j)}\frac{1}{z-v_j},$$

where we assume that $\gamma_{n+k} = \gamma_k$, $b_{n+k} = b_k$. The proof of Theorem 3 in this case will be concluded once we check the following:

$$\gamma_{k+1} - \frac{\gamma_k}{f'(b_k)} + \frac{f"(b_k)}{(f'(b_k))^2} = 0.$$

One can assume $k = 1$. We have:

$$\frac{\gamma_1}{f'(b_1)} - \frac{f"(b_1)}{(f'(b_1))^2} = \frac{(f^n)"(b_1)f'(b_1) - f"(b_1)\rho(1-\rho)}{(f'(b_1))^2\rho(1-\rho)}.$$

Now,

$$(f^n)"(b_1)f'(b_1) - f"(b_1)\rho(1-\rho) = f'(b_1)(f"(b_1)\Pi_{j=2}^n f'(b_j) +$$

$$\sum_{k=2}^n f"(b_k)((f^{k-1})'(b_1))^2 \Pi_{j=k+1}^n f'(b_j)) - f"(b_1)(\Pi_{j=1}^n f'(b_j) - \Pi_{j=1}^n (f'(b_j))^2) =$$

$$f'(b_1)\sum_{k=2}^n f"(b_k)((f^{k-1})'(b_1))^2 \Pi_{j=k+1}^n f'(b_j) - f"(b_1)(\Pi_{j=1}^n f'(b_j) - \Pi_{j=1}^n f'(b_j) - \Pi_{j=1}^n (f'(b_j))^2) =$$

$$f'(b_1)\sum_{k=2}^n f"(b_k)((f^{k-1})'(b_1))^2 \Pi_{j=k+1}^n f'(b_j) + f"(b_1)\Pi_{j=1}^n (f'(b_j))^2 = (f^n)"(b_2)(f'(b_1))^2.$$

Multiple critical points. Let a rational function f be such that $f(z) = \sigma z^{m_\infty} + ...$ at ∞. Suppose f has a critical point c with multiplicity $m > 1$. This means that $f'(z) = (z-c)^m Q(z)$, where Q is a rational function, such that $Q(c) \neq 0$. Let us approximate f by a sequence of rational functions f_n, such that $f_n(z) = \sigma_n z^{m_\infty} + ...$ at ∞, and so that all critical points of every f_n are simple. In particular, there are m critical points $c_1(n), ..., c_m(n)$ of f_n, such that $c_j(n) \to c$ as $n \to \infty$, for each $j = 1, ..., m$. For every n, f_n has a periodic orbit O_n, so that $O_n \to O$ as $n \to \infty$. Denote A_n the function corresponding to O_n and f_n. Then

$$A_n(z) - T_{f_n} A_n(z) \to A(z) - TA(z)$$

as $n \to \infty$. Thus according to the proven case of simple critical points the proof of the identity will be done if we show that the following limit exists:

$$\lim_{n\to\infty} \sum_{j=1}^m \frac{A_n(c_j(n))}{f_n"(c_j(n))} \frac{1}{f_n(c_j(n)) - z} = \frac{-L}{f(c) - z}, \tag{30}$$

for some L and all z with large modulus. We use again the Residue theorem. Fix a small circle C around c. Then, for every big n and $|z|$ large enough,

$$\lim_{n\to\infty} \frac{A_n(c_j(n))}{f_n"(c_j(n))} \frac{1}{f_n(c_j(n)) - z} = \lim_{n\to\infty} \frac{1}{2\pi i} \int_C \frac{A_n(w)}{f_n'(w)(f_n(w) - z)} dw = \frac{1}{2\pi i} \int_C \frac{A(w)}{f'(w)(f(w) - z)} dw.$$

An easy calculation shows that

$$\frac{A(w)}{f'(w)(f(w) - z)} = \frac{A(w)}{(w-c)^m Q(w)((f(c) - z) + O((w-c)^{m+1}))} =$$

$$\frac{A(w)}{Q(w)(f(c) - z)} \frac{1}{(w-c)^m} + O(w-c) = \sum_{k=0}^\infty \frac{B_k}{f(c) - z}(w-c)^{k-m} + O(w-c),$$

where B_k are defined by the Taylor expansion $A(w)/Q(w) = \sum_{k=0}^\infty B_k (w-c)^k$. It gives us (30) with

$$-L = B_{m-1} = \frac{1}{(m-1)!} \frac{d^{m-1}}{dw^{m-1}}|_{w=c}(\frac{A(w)}{Q(w)}).$$

3.4 Proof of Theorem 4

Beltrami coefficients. As it has been mentioned already, we derive this theorem by making use of quasiconformal deformations. The following fundamental facts about quasiconformal maps are well-known (see e.g. [1]) and will be used throughout the paper. The Measurable Riemann Theorem states that, given a Beltrami coefficient ν, there exists a unique quasiconformal homeomorphism ψ^ν of the plane with the complex dilatation ν, i.e., $\nu(z) = \frac{\partial \psi^\nu}{\partial \bar{z}} / \frac{\partial \psi^\nu}{\partial z}$ a.e., and such that ψ^ν fixes $0, 1$, and ∞. Assume that ν_t depends on a complex parameter t and

$$\nu_t(z) = t\mu(z) + t\epsilon(z,t), \tag{31}$$

where μ, ϵ belong to L_∞, and $||\epsilon(z,t)||_\infty \to 0$ as $t \to 0$. Then there exists

$$\frac{\partial \psi^{\nu_t}}{\partial t}|_{t=0}(z) = -\frac{1}{\pi} \int_{\mathbf{C}} \mu(w) \frac{z(z-1)}{w(w-1)(w-z)} d\sigma_w. \tag{32}$$

Let $f \in \Pi_{d,\bar{p}}$, and satisfy the conditions (1)-(2) of Sect. 3.2. Remember that $\nu(z,t)$ is an analytic family of invariant Beltrami coefficients on $\mathbf{C}$, such that $\nu(z,0) = 0$ and $\nu(z,t) = 0$ for z in the basin of infinity of f. Since $\nu(z,t)$ is differentiable at $t = 0$, $\nu(z,t) = t\mu(z) + t\epsilon(z,t)$, where $||\epsilon(z,t)||_\infty \to 0$ as $t \to 0$. Note that μ is invariant by f, too. Indeed, as $\nu(z,t)$ is f-invariant, for every t,

$$t\mu(f(z)) \frac{|f'(z)|^2}{f'(z)^2} + t\epsilon(t, f(z)) \frac{|f'(z)|^2}{f'(z)^2} = t\mu(z) + t\epsilon(z,t),$$

which, together with $\epsilon(z,t) \to 0$ for $t \to 0$, implies that $(|f'|/f')^2 \mu \circ f = \mu$. It follows similarly that μ vanishes at the basin of ∞, too (for $|t|$ small enough).

Let h_t be an analytic family of quasiconformal homeomorphism in the plane tangent to ∞, so that h_t has the complex dilatation $\nu(z,t)$, and $h_0 = id$. Then $f_t = h_t \circ f \circ h_t^{-1}$ is a family of polynomials from $\Pi_{d,\bar{p}}$, which is analytic in t. Denote by $O_t = h_t(O)$ the corresponding attracting periodic orbit of f_t, by $\rho(t)$ its multiplier, and by $v_j(t) = h_t(v_j)$ the critical values of f_t.

Speed of the multiplier. We need a formula for the speed of the multiplier of a periodic orbit in an analytic family of maps obtained by a quasiconformal deformation. A similar formula is proved in [16]. For completeness, we reproduce the proof here. It is based on the formula (32). Let $b \in O$. A fundamental region C near b is a (measurable) set, such that every orbit of the dynamics $z \mapsto f^n(z)$ near b enters C once. Note that then $f^{kn}(C)$, $k = 1, 2, ...,$ are again fundamental regions (tending to b). Usually, C will be a domain bounded by a small simple curve that surrounds b and its image by f^n.

Lemma 3.2

$$\frac{\rho'(0)}{\rho} = -\frac{1}{\pi} \lim_{C \to \{b\}} \int_C \frac{\mu(z)}{(z-b)^2} d\sigma_z, \tag{33}$$

where C is a fundamental region near b.

Proof. Let us linearize f^n near b by fixing a disk $D = \{|z| < r_0\}$ and a univalent map $K : D \to \mathbf{C}$, such that $K(0) = b$ and $f^n \circ K = K \circ \rho$ in D. Given a function τ defined near the point b denote by $\hat{\tau} = |K'|^2/(K')^2 \tau \circ K$ the pullback of τ to D. For every t, the pullback $\hat{\nu}(w,t)$ of the Beltrami coefficient $\nu(w,t)$ is invariant by the linear map $\rho : w \mapsto \rho w$, i.e., $\hat{\nu}(w,t) = |\rho|^2/\rho^2 \hat{\nu}(\rho w, t)$. For every fixed t, extend $\hat{\nu}(w,t)$ to $\mathbf{C}$ by the latter equation. Denote by ϕ_t the quasiconformal map of the plane with the complex dilatation $\hat{\nu}(w,t)$, which fixes $0, 1$, and ∞. Then the map $\phi_t \circ \rho \circ \phi_t^{-1}$ is again linear $w \mapsto \lambda(t) w$, for some $|\lambda(t)| < 1$. It is easy to see from the construction that $w \mapsto \lambda(t) w$ is analytically conjugate to f_t^n near $h_t(b)$. Therefore, $\lambda(t) = \rho(t)$. Note that $\hat{\nu}(w,t) = t\hat{\mu}(w) + t\hat{\epsilon}(w,t)$, where $||\hat{\epsilon}(w,t)||_\infty \to 0$ as $t \to 0$. In particular, $\hat{\mu}$ is invariant by the linear map ρ, too. By the change of coordinates $z = K(w)$, the formula (33) reads now:

$$\frac{\rho'(0)}{\rho} = -\frac{1}{\pi} \int_{\hat{C}} \frac{\hat{\mu}(w)}{w^2} d\sigma_w, \tag{34}$$

where $\hat{C}$ is a fundamental region of $w \mapsto \rho w$. We prove the latter formula. From the invariance of $\hat{\mu}$, one can assume that $\hat{C} = \{w : |\rho| < |w| < 1\}$. Differentiating the equation $\rho(t)\phi_t(w) = \phi_t(\rho w)$ by t at $t = 0$, we get, for $w \neq 0$:

$$\rho'(0) = \frac{1}{w}\left(\frac{\partial}{\partial t}|_{t=0}\phi_t(\rho w) - \rho \frac{\partial}{\partial t}|_{t=0}\phi_t(w)\right),$$

where, by (32),

$$\frac{\partial}{\partial t}|_{t=0}\phi_t(w) = -\frac{1}{\pi}\int_{\mathbf{C}} \hat{\mu}(u) \frac{w(w-1)}{u(u-1)(u-w)} d\sigma_u.$$

After elementary transformations and using the invariance of $\hat{\mu}$, we get

$$\rho'(0) = -\frac{\rho(\rho-1)}{\pi} w \sum_{n \in \mathbf{Z}} \int_{\hat{C}} \frac{\hat{\mu}(\rho^n z)|\rho|^{2n}}{\rho^n z(\rho^n z - \rho w)(\rho^n z - w)} d\sigma_z =$$

$$= -\frac{\rho}{\pi}\int_{\hat{C}} \frac{\hat{\mu}(z)}{z} \lim_{N \to +\infty} \sum_{n=-N}^{N} \left(\frac{\rho^{n-1}}{\rho^{n-1}z - w} - \frac{\rho^n}{\rho^n z - w}\right) d\sigma_z =$$

$$= -\frac{\rho}{\pi}\int_{\hat{C}} \frac{\hat{\mu}(z)}{z} \lim_{N \to +\infty} \left(\frac{\rho^{-N-1}}{\rho^{-N-1}z - w} - \frac{\rho^N}{\rho^N z - w}\right) d\sigma_z = -\frac{\rho}{\pi}\int_{\hat{C}} \frac{\hat{\mu}(z)}{z^2} d\sigma_z,$$

because $|\rho| < 1$.

$\square$

Adjoint identity. We want to integrate the identity

$$A(z) - (TA)(z) = \sum_{j=1}^{p} \frac{L_j}{z - v_j}. \tag{35}$$

against the f-invariant Beltrami form μ. Recall that μ vanishes in a neighborhood of ∞, so that $\mu A = 0$ there. On the other hand, A is not integrable at the points of the periodic orbit O. To deal with this situation, for every small $r > 0$, consider the domain V_r to be the plane $\mathbf{C}$ with the following sets deleted: $B(b_1, r)$ union with $f_{b_{n-k+1}}^{-k}(B(b_1, r))$, for $k = 1, ..., n-1$, where $f_{b_{n-k+1}}^{-k}$ is a local branch of f^{-k} taking $b_1 \in O$ to b_{n-k+1}. In other words,

$$V_r = \mathbf{C} \setminus \{B(b_1, r) \cup_{k=1}^{n-1} f_{b_{n-k+1}}^{-k}(B(b_1, r))\}.$$

Then A is integrable in V_r, and, therefore, by the invariance of μ,

$$\int_{V_r} TA(z)\mu(z) d\sigma_z = \int_{f^{-1}(V_r)} A(z)\mu(z).$$

Now, $f^{-1}(V_r) = V_r \setminus (C_r \cup \Delta_r)$, where $C_r = f_{b_1}^{-n}(B(b_1, r)) \setminus B(b_1, r)$ is a fundamental region near b_1 defined by the local branch $f_{b_1}^{-n}$ that fixes b_1, and, in turn, Δ_r is an open set which is away from O and shrinks to a finitely many points as $r \to 0$. Therefore,

$$\int_{V_r} (A(z) - TA(z))\mu(z) d\sigma_z = \int_{C_r} A(z)\mu(z) d\sigma_z + o_r(1) \tag{36}$$

where $o(1) \to 0$ as $r \to 0$. It is easy to see that

$$\int_{C_r} A(z)\mu(z) d\sigma_z = \int_{C_r} \frac{\mu(z)}{(z - b_1)^2} d\sigma_z + o(1).$$

Thus,

$$\int_{V_r} (A(z) - TA(z))\mu(z) d\sigma_z = \int_{C_r} \frac{\mu(z)}{(z - b_1)^2} d\sigma_z + o(1). \tag{37}$$

The identity (35) then gives us:

$$\int_{C_r} \frac{\mu(z)}{(z - b_1)^2} d\sigma_z + o(1) = \sum_{j=1}^{p} L_j \int_{V_r} \frac{\mu(z)}{z - v_j} d\sigma_z \tag{38}$$

Lemma 3.2 allows us to pass to the limit as $r \to 0$:

$$\frac{\rho'(0)}{\rho} = -\frac{1}{\pi} \sum_{j=1}^{p} L_j \int_{\mathbf{C}} \frac{\mu(z)}{z - v_j} d\sigma_z. \tag{39}$$

Speed of critical values. Now we want to express the integral of $\mu(z)/(z-v_j)$ via $v_j'(0)$. It follows from $v_j(t) = h_t(v_j)$, that

$$v_j'(0) = \frac{\partial h_t}{\partial t}|_{t=0}(v_j). \tag{40}$$

Let ψ_t be the quasiconformal homeomorphism of the plane with the complex dilatation $\nu(z,t)$, that fixes $0, 1$ and ∞. Since h_t has the same complex dilatation and fixes ∞ too, we have: $h_t = a(t)\psi_t + b(t)$, where a, b are analytic in t (because h_t is so), and $a(0) = 1$, $b(0) = 0$, because $h_0 = id$. Then

$$\frac{\partial h_t}{\partial t}|_{t=0}(z) = a'(0)z + b'(0) + \kappa(z) \tag{41}$$

where

$$\kappa(z) = \frac{\partial \psi_t}{\partial t}|_{t=0}(z) = -\frac{1}{\pi}\int_{\mathbf{C}} \mu(w)\frac{z(z-1)}{w(w-1)(w-z)}d\sigma_w. \tag{42}$$

In other words,

$$\frac{\partial h_t}{\partial t}|_{t=0}(z) = z(a'(0)+\frac{1}{\pi}\int_{\mathbf{C}}\frac{\mu(w)}{w(w-1)}d\sigma_w)+b'(0)+\frac{1}{\pi}\int_{\mathbf{C}}\frac{\mu(w)}{w}d\sigma_w-\frac{1}{\pi}\int_{\mathbf{C}}\frac{\mu(w)}{w-z}d\sigma_w, \tag{43}$$

where the integrals exist because μ vanishes near ∞.

On the other hand, since f stays in the space of centered monic polynomials, we come to the following normalization at ∞: $h_t(z) = z + O(1/z)$. As $h_t(z)$ is holomorphic in t and z for $|t|$ small and $|z|$ big, we conclude from this that $(\partial h_t/\partial t)|_{t=0}(z) = O(1/z)$. Coming back to (43) we see that it is possible if and only if

$$\frac{\partial h_t}{\partial t}|_{t=0}(z) = -\frac{1}{\pi}\int_{\mathbf{C}}\frac{\mu(w)}{w-z}d\sigma_w. \tag{44}$$

In particular,

$$v_j'(0) = -\frac{1}{\pi}\int_{\mathbf{C}}\frac{\mu(w)}{w-v_j}d\sigma_w.$$

Plugging this in (39), we get finally:

$$\frac{\rho'(0)}{\rho} = \sum_{j=1}^{p} L_j v_j'(0). \tag{45}$$

This proves Theorem 4. According to the Concluding argument of Sect 3.2, Theorem 4 yields Theorem 1.

4 Proof of Theorem 2

Assume the contrary: the rank of the matrix **O** is less than r. Then its rows are linearly dependent. Let us write down the connection (10) for every periodic orbit O_j. We introduce some notations: $O_j = \{b_k^j\}_{k=1}^{n_j}$ the set of points of the periodic orbit O_j of period n_j, and the function $\tilde{B}_j$ is said to be B_{O_j} iff $\rho_j \neq 1$ and $\hat{B}_{O_j}$ iff $\rho_j = 1$. Remember that

$$B_{O_j}(z) = \sum_{k=1}^{n_j} \{ \frac{\rho_j}{(z - b_k^j)^2} + \frac{1}{1 - \rho_j} \frac{(f^{n_j})"(b_k^j)}{z - b_k^j} \}, \tag{46}$$

and

$$\hat{B}_{O_j}(z) = \sum_{k=1}^{n_j} \frac{(f^{n_j})"(b_k^j)}{z - b_k^j}. \tag{47}$$

First of all, we observe that each B_j is not identically zero. Indeed, this is obvious if $\rho_j \neq 0$ and $\rho_j \neq 1$. But if $\rho_j = 0$, then, by the assumption of the theorem, there is precisely one critical point c among the points of O_j, and $f"(c) \neq 0$. One can assume $b_1^j = c$. Then $(f^{n_j})"(b_1^j) = f"(c) \Pi_{k=2}^{n_j} f'(b_k^j) \neq 0$. This guarantees that B_j is not zero in this case, too. If $\rho_j = 1$, then, by the assumption, $(f^{n_j})"(b_k^j) \neq 0$, and hence $\hat{B}_j$ is not zero in this case as well. In this notation, the connections (10), (17) read as follows: $\tilde{B}_j(z) - (T\tilde{B}_j)(z) = \sum_{i=1}^{q} \frac{\tilde{\partial}^V \rho_j}{\partial V_i} \frac{1}{z - V_i}$, for every $j = 1, ..., r$. Now, the assumption implies that there exists a linear combination L of $\tilde{B}_j$, $j = 1, ..., r$, such that

$$L(z) - (TL)(z) = 0. \tag{48}$$

Since no function $\tilde{B}_j$ is zero and the periodic orbits O_j are different, L is a non-zero rational function with (possible) double poles at the points of the periodic orbits O_j, $1 \leq j \leq r$. Let j' denote indexes corresponding to neutral periodic orbits O_j (if any). Given $r > 0$ small enough, we define a domain $V_{r,R}$ as the plane with the following sets taken away: (i) the neighborhood $B^*(R)$ of ∞, (ii) the basin of attraction of O_j provided that O_j is attractive, (iii) if O_j is neutral, then the set (to be deleted) is the disk $B(b_1^j, r)$ union with $f_j^{-k}(B(b_1^j, r))$, for $1 \leq k \leq n_j - 1$, where f_j^{-k} is a local branch of f^{-k} taking $b_1^j \in O_j$ to $b_{n_j-k+1}^j$.

Then L is integrable in $V_{r,R}$. We fix R large enough. For any fixed parameter λ, such that $0 < \lambda < 1$, and $r > 0$ small enough,

$$f^{-1}(V_{r,R}) \subset \{V_{r,R} \setminus (f^{-1}(B^*(R)) \setminus B^*(R))\} \cup \cup_{j'} \{B(b_1^{j'}, r) \setminus B(b_1^{j'}, \lambda r)\}. \tag{49}$$

Note that $\int_{B(b,r) \setminus B(b, \lambda r)} \frac{1}{|z-b|^2} d\sigma_z \to 2\pi \log \lambda^{-1}$ as $r \to 0$. On the other hand, (48) implies that

$$0 = \int_{V_{r,R}} |L - TL| d\sigma_z \geq \int_{V_{r,R}} |L| d\sigma_z - \int_{V_{r,R}} |TL| d\sigma_z \geq \int_{V_{r,R}} |L| d\sigma_z - \int_{f^{-1}V_{r,R}} |L| d\sigma_z.$$

As $r \to 0$, we then get $\int_{f^{-1}(B^*(R))\setminus B^*(R)} |L(z)| d\sigma_z \le \sum_{j'} C_{j'} \log \lambda^{-1}$, with some $C_{j'} \ge 0$, which is impossible if R is fixed and λ is close enough to 1, which contradicts the assumption.

Comment 9 *This same proof shows the classical bound $r \le q$ (see Comment 5). Indeed, otherwise the rows of $\mathbf{O}$ are again linearly dependent, and the proof above applies. See also Comment 11 for rational functions and some further discussions.*

5 Rational maps. Main results

5.1 Spaces associated to a rational map

Similar to the polynomial case, let us introduce a space $\Lambda_{d,\bar{p}'}$ of rational functions and its subspace $\Lambda^{q'}_{d,\bar{p}'}$ as follows.

Definition 5.1 *Let $d \ge 2$ be an integer, and $\bar{p}' = \{m_j\}_{j=1}^{p'}$ a set of p' positive integers, such that $\sum_{j=1}^{p'} m_j = 2d-2$. A rational function f of degree d belongs to $\Lambda_{d,\bar{p}'}$ if and only if it satisfies the following conditions:*

(1) Infinity is a simple fixed point of f; more precisely,

$$f(z) = \sigma z + m + \frac{P(z)}{Q(z)}, \tag{50}$$

where $\sigma \ne 0, \infty$, and Q, P are polynomials of degrees $d-1$ and at most $d-2$ resp., which have no common roots. Without loss of generality, one can assume that $Q(z) = z^{d-1} + a_1 z^{d-2} + ... + a_{d-1}$ and $P(z) = b_0 z^{d-2} + ... + b_{d-2}$,

(2) f has precisely p' geometrically different critical points $c_1, ..., c_{p'}$, and the multiplicity of c_j is equal to m_j, that is, the equation $f(w) = z$ has precisely $m_j + 1$ different solutions for z near c_j and $z \ne c_j$, $j = 1, ..., p'$. Denote by $v_1, ..., v_{p'}$, $v_j = f(c_j)$, corresponding critical values. We assume that some of them can coincide as well as some can be ∞. By $p = p_f$ we denote usually the number of critical points of f with finite images, i.e. so that the corresponding critical values are finite. By definition, $p < p'$ if and only if infinity is a critical value.

The space $\Lambda^{q'}_{d,\bar{p}'}$, for some $1 \le q' \le p'$, consists of those $f \in \Lambda_{d,\bar{p}'}$, for which f has precisely q' geometrically different critical values, i.e., the set $\{v_j = f(c_j),\ j = 1, ..., p'\}$ contains q' different points (including possibly infinity). If ∞ is a critical value, then f has $q = q' - 1$ different finite critical values.

Finally, we define the space S_d as follows. Consider first $\Lambda_{d,\overline{2d-2}}$, in other words, the space of maps with simple critical points. Now, S_d is said to be its subspace consisting of maps f, such that every critical value of f is finite.

By a Mobius change of coordinate, every rational function f of degree $d \ge 2$ belongs to some $\Lambda_{d,\bar{p}'}$. Indeed, f has either a repelling fixed point, or a fixed point

with the multiplier 1 (see e.g. [24]). Hence, there exists a Mobius transformation M, such that ∞ is a fixed non-attracting point of $\tilde{f} = M \circ f \circ M^{-1}$.

Let us identify $f \in \Delta_{d,\bar{p}'}$ as above with the point

$$\bar{f} = \{\sigma, m, a_1, ..., a_{d-1}, b_0, ..., b_{d-2}\}$$

of $\mathbf{C}^{2d}$. It defines an analytic (in fact, algebraic) variety in $\mathbf{C}^{2d}$. We denote it again by $\Lambda_{d,\bar{p}'}$. We will see that $\Lambda_{d,\bar{p}'}$ has a natural structure of a manifold of complex dimension $p' + 2$, see Sect. 5.2.

The set $\Lambda_{d,\bar{p}'}$ is connected. Apparently, this follows from [21] although we will *not* use this non-trivial statement in the paper. On the other hand, we will need a much easier fact: the space S_d is path-connected. This will be used in the proof precisely like the path-connectedness of the space $\Pi_{d,\bar{p}}$ is used in the polynomial case. To show the path-connectedness of S_d, we proceed as follows. For any two rational functions $f_i(z) = P_i(z)/Q_i(z)$, $i = 1, 2$, let us define $[f_1, f_2](\gamma) := ((1-\gamma)P_1 + tP_2)/((1-\gamma)Q_1 + \gamma Q_2)$, for $\gamma \in \mathbf{C}$. It is easy to see that, except for finitely many γ's, $[f_1, f_2](\gamma) \in S_d$ provided $f_1, f_2 \in S_d$. Now, choosing a path $\gamma : [0, 1] \to \mathbf{C}$ avoiding exceptional γ's, we get a path in S_d that joins their arbitrary two points f_1, f_2.

5.2 Local coordinates

We introduce what is going to be a local coordinate $\bar{v}(f)$ of f in $\Lambda_{d,\bar{p}'}$. For a rational function $f \in \Lambda_{d,\bar{p}'}$, by $c_j(f)$ and $v_j(f) = f(c_j(f))$ we denote its critical points and critical values resp., and by $\sigma(f)$, $m(f)$ the corresponding data at ∞, so that $f(z) = \sigma(f)z + m(f) + O(1/z)$. Now, fix $f_0 \in \Lambda_{d,\bar{p}'}$, and consider maps f in a small enough neighborhood of $\bar{f}_0$ in $\Lambda_{d,\bar{p}'}$. Introduce a vector $\bar{v}(f) \in \mathbf{C}^{p'+2}$ as follows. Let us fix an order $c_1(f_0), ..., c_{p'}(f_0)$ in the collection of all critical points of f_0. Moreover, we will do it in such a way, that: (a) first p indexes correspond to finite critical values, i. e. $v_j(f_0) \neq \infty$ for $1 \le j \le p$ and $v_j(f_0) = \infty$ for $p < j \le p'$ (if $p < p'$), (b) if $v_i(f_0) = v_j(f_0)$, then $v_i(f_0) = v_k(f_0)$, for $i \le k \le j$. There exist p' functions $c_1(f), ..., c_{p'}(f)$, which are defined and continuous in a small neighborhood of f_0 in $\Lambda_{d,\bar{p}'}$, such that they constitute all different critical points of f of the multiplicities m_j. Define now the vector $\bar{v}(f)$. If all critical values of f_0 are finite, then we set

$$\bar{v}(f) = \{\sigma(f), m(f), v_1(f), ..., v_{p'}(f)\},$$

with the order from above. If some of the critical values $v_j(f_0)$ of f_0 are infinity, that is, $v_j(f_0) = \infty$ for $p < j \le p'$, then we replace in the definition of $\bar{v}(f)$ corresponding $v_j(f)$ by their reciprocals $v_j(f)^{-1}$:

$$\bar{v}(f) = \{\sigma(f), m(f), v_1(f), ..., v_p(f), v_{p+1}(f)^{-1}, ..., v_{p'}(f)^{-1}\}.$$

In particular, $\bar{v}(f_0) = \{\sigma(f_0), m(f_0), v_1(f_0), ..., v_p(f_0), 0, ..., 0\}$.

The function $f_0 \in \Lambda_{d,\bar{p}'}$ belongs to a unique subspace $\Lambda^{q'}_{d,\bar{p}'}$. Then, for $f \in \Lambda^{q'}_{d,\bar{p}'}$ close to f_0, we define another vector $\bar{V}(f) \in \mathbf{C}^{q'+2}$ by retaining each critical value in $\bar{v}(f)$ once.

We have a local map $\delta : f \mapsto \bar{v}(f)$ from a neighborhood of f_0 in the space $\Lambda_{d,\bar{p}'}$ to $\mathbf{C}^{p'+2}$.

Proposition 3 *The map*

$$\delta : \Lambda_{d,\bar{p}'} \to \mathbf{C}^{p'+2}$$

is locally a biholomorphic isomorphism between some neighborhoods of $\bar{f}_0 \in \Lambda_{d,\bar{p}'}$ and $\bar{v}(f_0) \in \mathbf{C}^{p'+2}$. In particular, $\Lambda_{d,\bar{p}'}$ is a manifold of dimension $p' + 2$.

In Sect. 7 we give two proofs of this basic fact.

The space $\Lambda_{d,\bar{p}'}$ can therefore be identified in a neighborhood of its point f_0 with a neighborhood W_{f_0} of $\bar{v}(f_0) \in \mathbf{C}^{p'+2}$. If, moreover, $f_0 \in \Lambda^{q'}_{d,\bar{p}'}$, for some q', then its neighborhood in $\Lambda^{q'}_{d,\bar{p}'}$ is the intersection of W_{f_0} with a q'-dimensional linear subspace of $\mathbf{C}^{p'+2}$ defined by the conditions: $v_i = v_j$ iff $v_i(f_0) = v_j(f_0)$. Thus the vector $\bar{V}(f)$ serves as a local coordinate system in $\Lambda^{q'}_{d,\bar{p}'}$.

5.3 Connection between the dynamics and parameters: the main formula

Let f be a rational function. Suppose $f \in \Lambda_{d,\bar{p}'}$, and, moreover, $f \in \Lambda^{q'}_{d,\bar{p}'}$. These will be the parameter spaces associated to f. Consider any finite *periodic orbit* $O = \{b_k\}_{k=1}^n$ of f of exact period n, with the multiplier $\rho = (f^n)'(b_k) = \Pi_{j=1}^n f'(b_j) \neq 1$. (For $\rho = 1$, see Subsect. 5.4.) By the Implicit Function theorem and by Proposition 3, there is a set of n functions $O(\overline{v}) = \{b_k(\overline{v})\}_{k=1}^n$ defined and holomorphic in $\overline{v} \in \mathbf{C}^{\mathbf{p}'+\mathbf{2}}$ in a neighborhood of $\bar{v}(f)$, such that $O(\overline{v}) = O$ for $\overline{v} = \overline{v}(f)$, and $O(\bar{v})$ is a periodic orbit of $g \in \Lambda_{d,\bar{p}}$ of period n, where g is in a neighborhood of f, and $\bar{v} = \bar{v}(g)$. In particular, if $\rho(\bar{v})$ denotes the multiplier of the periodic orbit $O(\bar{g})$ of g, it is a holomorphic function of $\bar{v}$ in this neighborhood. The standard notation $\partial\rho/\partial v_j$, $1 \leq j \leq p$, denotes the partial derivatives of ρ w.r.t the *finite* critical values of f.

Now, suppose that g stays in a neighborhood of f inside $\Lambda^{q'}_{d,\bar{p}'}$. Set $q = q'$ iff all critical values are finite, and $q = q' - 1$ otherwise. Then the multiplier ρ of $O(g)$ is, in fact, a holomorphic function of the vector of q different critical values $\{V_1, ..., V_q\}$ of g iff $q' = q$, i.e. they are all finite, and the vector $\{V_1, ..., V_q, 1/V_{q+1}\}$ iff $q' = q + 1$, i.e. f has an infinite critical value. By $\partial^V \rho/\partial V_k$ we then denote the partial derivatives of ρ w.r.t. the different *finite* critical values $V_1, ..., V_q$. We have: $\frac{\partial^V \rho}{\partial V_k} = \sum_{j:v_j=V_k} \frac{\partial \rho}{\partial v_j}$.

Theorem 5 *Suppose $f \in \Lambda_{d,\bar{p}'}$. Let O be a periodic orbit of f with multiplier $\rho \neq 1$, and $B = B_O$. Then*

$$B(z) - (TB)(z) = \sum_{j=1}^{p} \frac{\partial \rho}{\partial v_j} \frac{1}{z - v_j}, \tag{51}$$

where $v_j = v_j(f)$, $1 \leq l \leq p$, are all finite critical values corresponding to different critical points. We have:

for $1 \leq j \leq p$ (i.e. for finite critical values of f):

$$\frac{\partial \rho}{\partial v_j} = -\frac{1}{(m_j-1)!} \frac{d^{m_j-1}}{dw^{m_j-1}}|_{w=c_j} \frac{B(w)}{Q_j(w)} = -\frac{1}{2\pi i} \int_{|w-c_j|=r} \frac{B(w)}{f'(w)} dw, \tag{52}$$

for $p < j \leq p'$ (i.e. for infinite critical values of f):

$$\frac{\partial \rho}{\partial (v_j^{-1})} = \frac{1}{(m_j-1)!} \frac{d^{m_j-1}}{dw^{m_j-1}}|_{w=c_j} \frac{B(w)}{Q_j(w)} = \frac{1}{2\pi i} \int_{|w-c_j|=r} \frac{B(w)}{(1/f)'(w)} dw, \tag{53}$$

where Q_j is defined by $f'(w) = (w-c_j)^{m_j} Q_j(w)$ for $1 \leq j \leq p$, and $(1/f)'(w) = (w-c_j)^{m_j} Q_j(w)$ for $p < j \leq p'$.

Also,

$$\frac{\partial \rho}{\partial \sigma} = \frac{\tilde{\Gamma}_2}{\sigma}, \qquad \frac{\partial \rho}{\partial m} = \frac{\tilde{\Gamma}_1}{\sigma}, \tag{54}$$

where $\tilde{\Gamma}_1$, $\tilde{\Gamma}_2$ are defined by the expansion

$$B(z) = \frac{\tilde{\Gamma}_1}{z} + \frac{\tilde{\Gamma}_2}{z^2} + O(\frac{1}{z^3})$$

at infinity:

$$\tilde{\Gamma}_1 = \frac{1}{1-\rho} \sum_{k=1}^{n} (f^n)"(b_k), \qquad \tilde{\Gamma}_2 = n\rho + \frac{1}{1-\rho} \sum_{k=1}^{n} b_k (f^n)"(b_k). \tag{55}$$

If $f \in \Lambda^{q'}_{d,\bar{p}'}$, then

$$B(z) - (TB)(z) = \sum_{k=1}^{q} \frac{\partial^V \rho}{\partial V_k} \frac{1}{z - V_k}, \tag{56}$$

where V_k, $k = 1, ..., q$, are all pairwise different and finite critical values of f.

The proof is very similar to the one for polynomials, and is based on the Teichmuller theory of rational maps. However, it is more technical, because of two extra parameters σ, m at ∞, see Sects. 6-10.

5.4 Cusps

Here we consider, very similar to the polynomial case, Subsect. 2.3, the remaining case $\rho = 1$, under the assumption that the periodic orbit $O = \{b_1, ..., b_n\}$ of f is *non-degenerate*: $(f^n)''(b_j) \neq 0$, for some, hence, for any $j = 1, ..., n$. Then, for any rational function g, which is close to f, the map g in a small neighborhood of O has either precisely two different periodic orbits $O_g^{\pm}$ of period n with multipliers $\rho^{\pm} \neq 1$, or precisely one periodic orbit O_g of period n with the multiplier 1.

Now, suppose $f \in \Lambda_{d,\bar{p}'}^{q'}$, and let f_i, $i = 1, 2, ...$, be any sequence of maps from $\Lambda_{d,\bar{p}'}$, such that $f_i \to f$, $i \to \infty$. We assume that each f_i has a periodic orbit O_i near O, such that its multiplier $\rho_i \neq 1$. In other words, O_i is one of the periodic orbits $O_{f_i}^{\pm}$. As in Subsect. 2.3, the sequence of functions $\hat{B}_i(z) = (1-\rho_i)B_{O_i}(z)$ tends, as $i \to \infty$, to the rational function $\hat{B}(z) = \sum_{b \in O} \frac{(f^n)''(b)}{z-b}$. As in Subsect. 2.3, Theorem 5 implies:

Proposition 4 *The following finite limits exist:*

$$\frac{\hat{\partial}\rho}{\partial v_j} := \lim_{i\to\infty}(1-\rho_i)\frac{\partial \rho_i}{\partial v_j}, \quad j = 1, ..., p, \tag{57}$$

$$\frac{\hat{\partial}\rho}{\partial \sigma} := \lim_{i\to\infty}(1-\rho_i)\frac{\partial \rho_i}{\partial \sigma}, \quad \frac{\hat{\partial}\rho}{\partial m} := \lim_{i\to\infty}(1-\rho_i)\frac{\partial \rho_i}{\partial m}. \tag{58}$$

Then we have:

$$\hat{B}(z) - (T\hat{B})(z) = \sum_{j=1}^{p} \frac{\hat{\partial}\rho}{\partial v_j}\frac{1}{z - v_j}. \tag{59}$$

The formula (52) holds, where one replaces ρ and B by $\hat{\rho}$ and $\hat{B}$ respectively. Also,

$$\frac{\hat{\partial}\rho}{\partial \sigma} = \frac{\hat{\Gamma}_2}{\sigma}, \qquad \frac{\hat{\partial}\rho}{\partial m} = \frac{\hat{\Gamma}_1}{\sigma}, \tag{60}$$

where $\hat{\Gamma}_1$, $\hat{\Gamma}_2$ are defined by the expansion

$$\hat{B}(z) = \frac{\hat{\Gamma}_1}{z} + \frac{\hat{\Gamma}_2}{z^2} + O(\frac{1}{z^3})$$

at infinity:

$$\hat{\Gamma}_1 = \sum_{k=1}^{n}(f^n)''(b_k), \qquad \hat{\Gamma}_2 = \sum_{k=1}^{n} b_k (f^n)''(b_k). \tag{61}$$

Furthermore, if f and f_i are in $\Lambda_{d,\bar{p}'}^{q'}$, then, for every $j = 1, ..., p$, there exists a finite limit

$$\frac{\partial^V \rho}{\partial v_j} := \lim_{i\to\infty}(1-\rho_i)\frac{\partial^V \rho_i}{\partial v_j}, \tag{62}$$

and

$$\hat{B}(z) - (T\hat{B})(z) = \sum_{k=1}^{q} \frac{\hat{\partial}^V \rho}{\partial V_k} \frac{1}{z - V_k}. \tag{63}$$

5.5 Multipliers and local coordinates

Multipliers versus critical values. Theorem 5, Proposition 4 and the contraction property of T yield the following.

Theorem 6 *Suppose that $f \in \Lambda^{q'}_{d,\bar{p}'}$, and let $V_1, ..., V_q$ be all the different and finite critical values of f. Suppose that f has a collection $O_1, ..., O_r$ of r different finite periodic orbits with the corresponding multipliers $\rho_1, ..., \rho_r$, such that each O_j is non-repelling: $|\rho_j| \leq 1$, $j = 1, ..., r$. Assume that, if, for some j, $\rho_j = 1$, then the periodic orbit O_j is non-degenerate. Denote by $\tilde{\partial}^V \rho_j / \partial V_k$ the $\partial^V \rho_j / \partial V_k$ iff $\rho_j \neq 1$ and $\hat{\partial}^V \rho_j / \partial V_k$ iff $\rho_j = 1$. Similar notation stands for $\tilde{\partial}^V \rho_j / \partial \sigma$.*

Assume also that if $\rho_j = 0$, then the periodic orbit O_j contains a single critical point, and it is simple.

(H_∞). If $\sigma \neq 1$ and $m = 0$, i.e., $f(z) = \sigma z + O(1/z)$ as $z \to \infty$, then, for every $1 \leq k \leq q$, such that $V_k \neq 0$, the rank of the following $q \times r$ matrix

$$\mathbf{O} = (\frac{\tilde{\partial} \rho_j}{\partial \sigma}, \frac{\tilde{\partial}^V \rho_j}{\partial V_1}, ..., \frac{\tilde{\partial}^V \rho_j}{\partial V_{k-1}}, \frac{\tilde{\partial}^V \rho_j}{\partial V_{k+1}}, ..., \frac{\tilde{\partial}^V \rho_j}{\partial V_q})_{1 \leq j \leq r} \tag{64}$$

is equal to r.

(H_∞^{attr}). If $\sigma \neq 1$ and $m = 0$, and, additionally, $|\sigma| \geq 1$, i.e., ∞ is either attracting or neutral fixed point, then, for every $1 \leq k \leq q$, such that $V_k \neq 0$, the rank of the following $q - 1 \times r$ matrix

$$\mathbf{O}^{\mathbf{attr}} = (\frac{\tilde{\partial}^V \rho_j}{\partial V_1}, ..., \frac{\tilde{\partial}^V \rho_j}{\partial V_{k-1}}, \frac{\tilde{\partial}^V \rho_j}{\partial V_{k+1}}, ..., \frac{\tilde{\partial}^V \rho_j}{\partial V_q})_{1 \leq j \leq r} \tag{65}$$

is equal to r.

(NN_∞). If $\sigma = 1$ and $m \neq 0$, then, for every $1 \leq k \leq q$, the rank of the following $q - 1 \times r$ matrix

$$\mathbf{O}^{\mathbf{neutral}} = (\frac{\tilde{\partial}^V \rho_j}{\partial V_1}, ..., \frac{\tilde{\partial}^V \rho_j}{\partial V_{k-1}}, \frac{\tilde{\partial}^V \rho_j}{\partial V_{k+1}}, ..., \frac{\tilde{\partial}^V \rho_j}{\partial V_q})_{1 \leq j \leq r} \tag{66}$$

is equal to r.

(ND_∞). Finally, if $\sigma = 1$ and $m = 0$, then, for every $1 \leq k < l \leq q$, the rank of the following $q - 2 \times r$ matrix

$$\mathbf{O}_{\mathbf{0}}^{\mathbf{neutral}} = (\frac{\tilde{\partial}^V \rho_j}{\partial V_1}, ..., \frac{\tilde{\partial}^V \rho_j}{\partial V_{k-1}}, \frac{\tilde{\partial}^V \rho_j}{\partial V_{k+1}}, ..., \frac{\tilde{\partial}^V \rho_j}{\partial V_{l-1}}, \frac{\tilde{\partial}^V \rho_j}{\partial V_{l+1}}, ..., \frac{\tilde{\partial}^V \rho_j}{\partial V_q})_{1 \leq j \leq r} \tag{67}$$

is equal to r.

For the proof, see Sect. 11.

Comment 10 *Note that $r \le q$ in the case (H_∞), $r \le q-1$ in the cases (H_∞^{attr}) and (NN_∞), and $r \le q-2$ in the case (ND_∞) (at least two attracting petals at infinity). These bounds follow from the Fatou-Shishikura inequality, see [34] and references therein. See also Comment 11.*

Moduli spaces. Here we discuss a moduli space of rational functions associated to a given one with respect to the standard equivalence relation. Then we apply Theorem 6. Most considerations in this paragraph are quite straightforward consequences of Proposition 3 and Theorem 6.

Suppose that f is an arbitrary rational function of degree $d \ge 2$. Denote by p' and q' respectively the number of different critical points and critical values of f in the Riemann sphere, and by $\bar{p}'$ the vector of multiplicities at the critical points. Introduce a space Rat^f of rational functions g of degree $d \ge 2$, such that f and g have the same (up to a permutation) vector $\bar{p}'$ of multiplicities at different critical points in the Riemann sphere, and the same number q' of different critical values. Define the moduli space Mod^f to be the quotient space $Mod^f = Rat^f/\sim$, where $f_1 \sim f_2$ iff f_1, f_2 are conjugated by a Mobius transformation. Denote by $[g]$ the equivalence class of $g \in Rat^f$. Clearly, Rat^f as well as Mod^f depend merely on $[f]$. Note that the multiplier of a periodic orbit of g is invariant by a holomorphic conjugation. Therefore, one can speak about the multiplier of a periodic orbit of the class $[g]$.

The rational function f has a fixed point, which is either repelling, or has the multiplier 1 (see e.g. [24]). Therefore, there is an alternative: either **(H)** f has a fixed point a, such that $f'(a) \neq 0, 1$, or **(N)** the multiplier of every fixed point of f is either 0 or 1, and there is a fixed point with the multiplier 1. The case **(N)** is degenerate. We consider each case separately and introduce a kind of cross-section in the moduli space near $[f]$.

(H). Let P be a Mobius transformation, such that $P(a) = \infty$. Then $\tilde{f} = P \circ f \circ P^{-1}$ belongs to $\Lambda_{d,\bar{p}'}^{q'}$. Moreover, P can be chosen uniquely in such a way, that one of the critical values of $\tilde{f}$ is equal to 1, and $m(\tilde{f}) = 0$, that is, $\tilde{f}(z) = \sigma z + O(1/z)$ at infinity. Let us define a submanifold $\Lambda_{\tilde{f}}$ of $\Lambda_{d,\bar{p}'}^{q'}$ consisting of $g \in \Lambda_{d,\bar{p}'}^{q'}$ in a neighborhood of $\tilde{f}$, such that $m(g) = 0$, and one of the critical values of g is identically equal to 1. Introduce the vector

$$\bar{V}_{\tilde{f}}(g) = \{\sigma(g), V_1(g), ..., V_{q'-2}(g), V^*_{q'-1}(g)\},$$

such that $V_1(g), ..., V_{q'-1}(g), 1$ are all different critical values of g, and $V_1(g), ..., V_{q'-2}(g)$ are finite while $V^*_{q'-1}(g) = V_{q'-1}(g)$ iff $V_{q'-1}(\tilde{f})$ is finite and $V^*_{q'-1}(g) = 1/V_{q'-1}(g)$ otherwise. We denote $q = q'$ in the former case, and $q = q'-1$ in the latter one. We see from Proposition 3, that $\bar{V}_{\tilde{f}}$ is a local coordinate of $\Lambda_{\tilde{f}}$: the correspondence

$g \in \Lambda_{\tilde{f}} \mapsto \bar{V}_{\tilde{f}}(g) \in \mathbf{C}^{q'}$ is biholomorphic from the manifold $\Lambda_{\tilde{f}}$ onto a neighborhood of the point $\bar{V}_{\tilde{f}}(\tilde{f})$ in $\mathbf{C}^{q'}$. Now we have a natural projection $[.]_V : \bar{V} \mapsto [g]$ from a neighborhood of $\bar{V}_{\tilde{f}}(\tilde{f}) \in \mathbf{C}^{\mathbf{q}'}$ into the space $Mod^{\tilde{f}}$, where $[\bar{V}]_V$ is said to be the equivalence class of the unique $g \in \Lambda_{\tilde{f}}$, such that $\bar{V}_{\tilde{f}}(g) = \bar{V}$.

(N). There are two subcases to distinguish.

(NN): f has a fixed point a, such that $f'(a) = 1$ and $f''(a) \neq 0$. Let P be a Mobius transformation, such that $P(a) = \infty$. Then $\tilde{f} = P \circ f \circ P^{-1}$ belongs to $\Lambda^{q'}_{d,\bar{p}'}$. Moreover, P can be chosen uniquely in such a way, that one of the critical values of $\tilde{f}$ is equal to 1, and $m(\tilde{f}) = 1$, that is, $\tilde{f}(z) = z + 1 + O(1/z)$ at infinity. Then we define $\Lambda_{\tilde{f}}$ to be the set of all $g \in \Lambda^{q'}_{d,\bar{p}'}$ in a neighborhood of $\tilde{f}$, such that $m(g) = 1$, and the critical value $V_{q'}(g)$ of g (which is close to $V_{q'}(\tilde{f}) = 1$) is identically equal to 1. The vector $\bar{V}_{\tilde{f}}$ is defined like in the previous case. It is a coordinate in $\Lambda_{\tilde{f}}$. As above, there is the projection $[.]_V : \bar{V} \mapsto [g]$ from a neighborhood of $\bar{V}_{\tilde{f}}(\tilde{f})$ in $\mathbf{C}^{\mathbf{q}'}$ into the space $Mod^{\tilde{f}}$.

(ND): every fixed point with multiplier 1 is degenerate. Let a be one of them: $f'(a) = 1$ and $f''(a) = 0$. Then the Mobius map P can be chosen uniquely in such a way, that $\tilde{f}(z) = P \circ f \circ P^{-1}(z) = z + O(1/z)$, and $\tilde{f}$ has a critical value equal to 1 in one attracting petal of ∞, and equal to 0 in another attracting petal of ∞. Then $\Lambda_{\tilde{f}}$ consists of $g \in \Lambda^{q'}_{d,\bar{p}'}$ in a neighborhood of $\tilde{f}$, such that the critical value of g, which is close to $V_{q'-1}(\tilde{f}) = 1$ is identically equal to 1, and the critical value of g, which is close to $V_{q'}(\tilde{f}) = 0$, is identically equal to 0. Define

$$\bar{V}_{\tilde{f}}(g) = \{\sigma(g), m(g), V_1(g), ..., V_{q'-3}(g), V^*_{q'-2}(g)\},$$

such that $V_1(g), ..., V_{q'-2}(g), 1, 0$ are all different critical values of g, and $V_1(g), ..., V_{q'-3}(g)$ are finite while $V^*_{q'-2}(g) = V_{q'-2}(g)$ iff $V_{q'-2}(\tilde{f})$ is finite and $V^*_{q'-2}(g) = 1/V_{q'-2}(g)$ otherwise. We denote $q = q' - 1$ in the former case, and $q = q' - 2$ in the latter one. We see from Proposition 3, that $\bar{V}_{\tilde{f}}$ is a local coordinate on $\Lambda_{\tilde{f}}$. There is the projection $[.]_V : \bar{V} \mapsto [g]$.

It is not hard to understand that the map $[.]_V$ sends a small neighborhood of the point $\bar{V}_{\tilde{f}}(\tilde{f})$ in $\mathbf{C}^{\mathbf{q}'}$ onto a neighborhood of the point $[\tilde{f}] = [f]$ in Mod^f. In fact, the map $[.]_V$ defines a complex q'-orbifold structure on Mod^f (see e.g. [30] for the definition of an orbifold).

Let us reformulate Theorem 6 for the map $[.]_V$. Suppose that $[f]$ has a collection of r different *non-repelling* periodic orbits with multipliers $\rho^0_1, ..., \rho^0_r$, i.e. $|\rho^0_j| \leq 1$. Assume additionally that $\rho^0_j \neq 1$, $j = 1, ..., r$, and, if $\rho^0_j = 0$, for some j, then the corresponding periodic orbit contains a single and simple critical point of the map. Let us consider a map $\tilde{f}$ corresponding to f, and fix an order $O_1, ..., O_r$ of the above periodic orbits of $\tilde{f}$. We then have a vector $(\rho^0_1, ..., \rho^0_r)$ of their multipliers. If the

map changes, the multipliers become functions $\rho_j(g)$ of the map g. In particular, $\rho_j(\tilde{f}) = \rho_j^0$. Then Theorem 6 implies the following:

Theorem 7 *There are r indexes $1 \le j_1 \le ... \le j_r \le q'$ as follows. One can replace the map $[.]_V$ by another map $[.]_\rho : \bar{V}_\rho \to Mod^{\tilde{f}}$ defined in a neighborhood of a point $\bar{V}_\rho(\tilde{f}) \in \mathbf{C}^{q'}$, where $\bar{V}_\rho$ is obtained from $\bar{V}_{\tilde{f}}$ by replacing the coordinates with indexes $j_1, ..., j_r$ by $\rho_{j_1}, ..., \rho_{j_r}$ respectively. The change of variables is biholomorphic. Moreover, in the case (H), if $|\sigma(\tilde{f})| \ge 1$, and in the case (NN) the above r coordinates in $\bar{V}_{\tilde{f}}$ can be chosen among the critical values $V_1, ..., V_{q'-1}$, while in the case (ND) they can be chosen among the critical values $V_1, ..., V_{q'-2}$.*

Let us consider a particular case of Theorem 7, which corresponds to maps with the maximal number of non-repelling periodic orbits. Namely, assume that the number of such orbits is $r = q = q'$. Then $[f]$ has necessarily a repelling fixed point, i.e. we are in the case (H). Therefore, the map $[.]_\rho$ depends on $\rho_1, ..., \rho_q$ only. It has an invariance property as follows. Suppose that $[\rho_1, ...\rho_q]_\rho = [\rho'_1, ..., \rho'_q]_\rho$, for two vectors of the multipliers $(\rho_1, ...\rho_q)$, $(\rho'_1, ...\rho'_q)$, which correspond to maps g, g' respectively. It is clear that then g and g' are conjugate by a Mobius transformation M. In turn, it defines a permutation π of $1, ..., q$ in the collection $O_1, ..., O_q$: M maps a periodic orbit $O_j(g)$ of g to the periodic orbit $O_{\pi(j)}(g')$ of g'. Then the invariance property is: $\rho'_{\pi(j)} = \rho_j$, $j = 1, ..., q$. It has an interesting consequence:

In the above set up $r = q = q'$, i.e., if the number of non-repelling periodic orbits with the multipliers different from 1 is equal to the number of different critical values, the map $[.]_\rho$ is locally injective in each coordinate ρ_j, $j = 1, ..., q$.

Indeed, if $[\rho_1, \rho_2, ...\rho_q]_\rho = [\rho'_1, \rho_2, ...\rho_q]_\rho$, then there is a permutation π as above. We have: if $\pi(1) = 1$, then $\rho'_1 = \rho_1$, and otherwise $\rho_1 = \rho_{\pi(1)} = \rho_{\pi^2(1)} = ... = \rho_{\pi^{l-1}(1)}$, where $l \ge 1$ is minimal so that $\pi^l(1) = 1$. But $\pi(\pi^{l-1}(1)) = 1$, hence, $\rho'_1 = \rho_{\pi^{l-1}(1)} = \rho_1$.

Let us find the degree of the map $[.]_V$. We fix the manifold $\Lambda_{\tilde{f}}$ in a small enough neighborhood of $\tilde{f}$ in such a way, that it is projected onto a neighborhood of $[f]$ in Mod^f. Denote by $|A|$ the number of points in a set A (a priori, $|A|$ could be infinite). Given a rational function g, we denote by $Aut(g)$ the finite group of Mobius transformations that commute with g. For $g \in \Lambda_{\tilde{f}}$, denote by $\mathbf{M(g)}$ the set of Mobius transformations M, such that $M^{-1} \circ g \circ M \in \Lambda^{\tilde{f}}$. Obviously, if $M \in \mathbf{M(g)}$ and $K \in Aut(g)$, then $K \circ M \in \mathbf{M(g)}$. It follows, $|Aut(g)|$ divides $|\mathbf{M(g)}|$. The quantity $|\mathbf{M(g)}|/|Aut(g)|$ is precisely the number of different maps $\psi \in \Lambda^{\tilde{f}}$, such that $\psi \in [g]$.

Claim. *Let $g \in \Lambda_{\tilde{f}}$. Then $|\mathbf{M(g)}|$ is finite and equal to*

$$|\mathbf{M(g)}| = D_g := \sum_{P \in Aut(\tilde{f})} L_g(P(\infty)),$$

where $L_g(Z)$ is the number of geometrically different fixed points of g near a fixed point Z of $\tilde{f}$. Consequently, the number of different maps $\psi \in \Lambda_{\tilde{f}}$, such that $\psi \in [g]$, is equal to $\frac{D_g}{|Aut(g)|}$.

We will not prove it here (and will not use it in the paper) although the consideration behind the proof is quite clear. Namely, for every $P \in Aut(\tilde{f})$ and every fixed point Z_g of g which is near $P(\infty)$ there is one and only one $M \in \mathbf{M(g)}$, such that $M(\infty) = Z_g$ and M is close to P.

We have the bound: $|\mathbf{M(g)}| \le |Aut(\tilde{f})|L_m$, where L_m is the maximal number of fixed points that can appear from the fixed point ∞ of the map $\tilde{f}$ under a perturbation of the map. Let us be more precise. For every $P \in Aut(\tilde{f})$, the point $P(\infty)$ is a fixed point of $\tilde{f}$ with the same multiplier as at ∞. We have: $1 \le L_g(P(\infty)) \le L_m$, where $L_m = 1$ in the case $\sigma(\tilde{f}) \neq 1$, and, if $\sigma(\tilde{f}) = 1$, the $L_m \ge 2$ is defined by: $\tilde{f}(z) = z + b/z^{L_m - 2} + ...$, $z \to \infty$, with $b \neq 0$. Let us call the number L_m the multiplicity of the fixed point. It is defined similarly for any fixed point with the multiplier 1.

Let us discuss briefly several cases. Assume first that $\sigma(\tilde{f}) \neq 1$. Then $|\mathbf{M(g)}| = |Aut(\tilde{f})|$ (hence, independent of $g \in \Lambda^{\tilde{f}}$). If, additionally, $Aut(f) = \{I\}$, then $[.]_V$ is injective. On the other hand, if $\sigma(\tilde{f}) = 1$, let us assume that g is non-degenerate, in a sense, that, firstly, $Aut(g)$ is trivial, and, secondly, $L_g(P(\infty)) = L_m$, i.e. g has the maximal number of fixed points near $P(\infty)$, for every $P \in Aut(\tilde{f})$. Then the number of different maps $\psi \in \Lambda_{\tilde{f}}$, such that $\psi \in [g]$, is maximal and equal to $|\mathbf{M(g)}| = |Aut(\tilde{f})|L_m$.

Let us come back to the general case. As g changes and different $\psi \in [g]$ from $\Lambda_{\tilde{f}}$ collide, this corresponds either to a collision of fixed points of g near some $P(\infty)$ or to the appearence of new maps in the group $Aut(g)$. To be more specific, given $g \in \Lambda_{\tilde{f}}$, define an equivalence relation in the set of all fixed points of g near the set $\bar{Z} = \{P(\infty)\}_{P \in Aut(\tilde{f})}$ as follows: two points $x, y \in \bar{Z}$ are equivalent if and only if there exists $K \in Aut(g)$ so that $y = K(x)$. To every equivalence class in $\bar{Z}$ there corresponds one and only one map $\psi \in [g]$, such that $\psi \in \Lambda_{\tilde{f}}$. We define the multiplicity of ψ as the sum of the multiplicities of all fixed points of g in this equivalence class (note that the multiplicities of all fixed points of the same class are equal). With this definition, we have (without any restriction on $\tilde{f}$): for every $g \in \Lambda_{\tilde{f}}$, the total number of $\psi \in [g]$ in $\Lambda_{\tilde{f}}$ each counted with its multiplicity is equal to $|Aut(\tilde{f})|L_m$.

Quadratic rational maps. For a degree two rational function f, $\bar{p}' = (1,1)$ and $q' = 2$, hence, $Rat_2 = Rat^f$ is the set of all quadratic rational maps, and $Mod_2 = Mod^f$ is the space of orbits of the quadratic maps by Mobius conjugations. It is easy to check that the degree of the map $[.]_V$ takes values $1, 2$, or 6. The spaces Rat_2 and Mod_2 have been studied intensively, see [31], [32], [33], [25]. Global

coordinates in Mod_2 are introduced in [25]. It turnes out Mod_2 is isomorphic to $\mathbf{C}^2$. The problem of multipliers as coordinates for hyperbolic (and some neutral) degree 2 rational maps is settled in [31]. Theorem 6 allows us to deal with not necessary hyperbolic maps. Let us state its corollary for degree two.

Suppose f is a rational function of degree 2 that has a periodic orbit O with the multiplier ρ, such that $|\rho| \le 1$. If $\rho = 0$, assume that the orbit O contains a single critical point. If $\rho = 1$, assume that O is not degenerate (i.e. each point of O has only one attracting petal) and, moreover, if f is conjugate to $z^2 + 1/4$, then O is not its neutral fixed point. Then, after a Mobius change of coordinates, $f(z) = \sigma z + m + O(1/z)$ and $O \neq \infty$, and also one of the (two) different critical values v_1, v_2 of f, say, v_2 is not zero. Moreover, if $\sigma \neq 1$, one can further assume that $m = 0$. As usual, the multiplier ρ is a function of σ, m, v_1, v_2 (for the moduli space, one can keep m and v_2 fixed though, see the general discussion above).

Corollary 5.1 *For $\rho \neq 1$, the vector $(\partial\rho/\partial\sigma, \partial\rho/\partial v_1)$ is not zero, and for $\rho = 1$, the vector $(\partial\hat{\rho}/\partial\sigma, \partial\hat{\rho}/\partial v_1)$ is not zero. Moreover, under the condition $|\sigma| \ge 1$ (i.e., the fixed point at ∞ is not repelling), we have:*

$$\partial\rho/\partial v_1 \neq 0$$

for $\rho \neq 1$, and

$$\partial\hat{\rho}/\partial v_1 \neq 0$$

for $\rho = 1$.

6 Theorem 5: an outline of the proof

First, we prove Theorem 5 for maps f in the space S_d, that is, assuming that every critical point of f is simple and every critical values is finite. It will occupy most of the rest of the paper. Then we prove Theorem 5 for multiple critical points still assuming that every critical value is finite. For this, we use a kind of a limit procedure, see Sect. 9. Finally, to complete the proof of Theorem 5, we send some of the critical values to ∞, see Sect. 10. So, assume (until Sect. 9) that $f \in S_d$. Since ρ is a holomorphic function in $\bar{v}$, it is enough to prove the formulae of Theorem 5 for $\rho \neq 0$.

The identity. We will use the same identity of Theorem 3.

Reduction to the hyperbolic case. Here we show that in order to prove Thoerem 5 for any $f \in S_d$, it is enough to prove it only for those f from S_d that satisfy the following conditions:

(1) f is a hyperbolic map, and ∞ is an attracting fixed point, i.e., $|\sigma| > 1$,

(2) f has no critical relations,

(3) O is an attracting periodic orbit of f.

Indeed, assume that Theorem 5 holds for this open subset of maps from S_d. Given now any $f \in S_d$ as in Theorem 5, we find a real analytic path g_t, $t \in [0,1]$, in S_d, which has the following properties: (i) $g_0 = f$, (ii) g_1 satisfies conditions (1)-(3), (iii) the analytic continuation O_t (a periodic orbit of g_t) of the periodic orbit O along the path is well-defined (i.e. the multiplier of O_t is not 1 for $t \in [0,1]$), and O_1 (the periodic orbit of g_1) is attracting.

Denote by $\Delta(z, \overline{f})$ the difference between the left and the right hand sides of (51). It is an analytic function in $\overline{f}$ in a neighborhood of every point $\bar{g}_t$, $t \in [0,1]$. On the other hand, by the assumption, it is identically zero in a neighborhood of $\bar{g}_1$. By the Uniqueness Theorem for analytic functions, $\Delta(z, \bar{f}) = 0$.

Let us show that the path g_t as above exists. We first connect f to the map $p_0 : z \mapsto z^d$ through a path γ_0 of the form $[f, p_0]$ (see Subsect. 5.1), so that the analytic continuation of the periodic orbit O of f along this path exists, and O turns into a periodic orbit Q of p_0. Then we proceed by a real analytic path c_Q in the parameter plane of $p_c(z) = z^d + c$ that turns Q into an attracting periodic orbit of some p_c. Finally, we find the desired path g_t in S_d in a neighborhood of $c_Q \circ \gamma_0$.

Hyperbolic maps Here we describe how to prove Theorem 5 for the maps *$f \in S_d$ that satisfy the conditions (1)-(3) of the previous paragraph.* Similar to the polynomial case, let $\nu(z,t)$ be an analytic family of invariant Beltrami coefficients in the Riemann sphere, and $\nu(z,0) = 0$. (We do not assume that $\nu(z,t) = 0$ for z near ∞ though.) In turn, let h_t be an analytic family of quasiconformal homeomorphism in the plane that fix ∞, so that h_t has the complex dilatation $\nu(z,t)$, and $h_0 = id$. Then $f_t = h_t \circ f \circ h_t^{-1}$ is an analytic family of rational functions. Moreover, $f_t \in S_d$, and $O_t = h_t(O)$ the corresponding attracting periodic orbit of f_t. Denote by $\rho(t)$ its multiplier, and by $v_j(t) = h_t(v_j)$ the set of finite critical values of f_t. Besides, $f_t(z) = \sigma(t)z + m(t)z + O(\frac{1}{z})$. Note that the functions $\sigma(t)$, $m(t)$, $\rho(t)$, and $v_j(t)$ are analytic in t, and $\sigma(0) = \sigma$, $m(0) = m$, $\rho(0) = \rho$, $v_j(0) = v_j$. Starting with Theorem 3, we derive:

Theorem 8 *For $f \in S_d$,*

$$\frac{\rho'(0)}{\rho} = \Gamma_2 \frac{\sigma'(0)}{\sigma} + \frac{\Gamma_1}{\sigma} m'(0) + \sum_{j=1}^{2d-2} L_j v_j'(0), \tag{68}$$

where Γ_1 and Γ_2 are defined by the expansion at infinity:

$$A(z) = \frac{\Gamma_1}{z} + \frac{\Gamma_2}{z^2} + O(\frac{1}{z^3}).$$

In the course of the proof we calculate $\sigma'(0)$ and $m'(0)$.

Concluding argument. We are going to compare (68) to the following obvious identity:

$$\rho'(0) = \frac{\partial\rho}{\partial\sigma}\sigma'(0) + \frac{\partial\rho}{\partial m}m'(0) + \sum_{j=1}^{2d-2}\frac{\partial\rho}{\partial v_j}v'_j(0). \qquad (69)$$

The proof will be finished once we will show that the vector

$$\{\sigma'(0), m'(0), v'_1(0), ..., v'_{2d-2}(0)\}$$

can be taken arbitrary in $\mathbf{C}^{2d}$. To this end, for every vector $\bar{v}' = \{\sigma', m', v'_1, ..., v'_{2d-2}\} \in \mathbf{C}^{2d}$ of initial conditions there exists an analytic family f_t of rational maps from S_d with the critical values $v_1(t), ..., v_{2d-2}(t)$, such that $f_t(z) = \sigma(t)z + m(t) + O(\frac{1}{z})$, and $\sigma'(0) = \sigma'$, $m'(0) = m'$, $v'_j(0) = v'_j$, for $1 \le j \le 2d-2$. Indeed, this is an immediate consequence of Proposition 3 for $f \in S_d$, where one can simply take locally $\bar{v}(t) = \bar{v} + t\bar{v}'$, and find by the inverse holomorphic correspondence $\bar{v} \mapsto \bar{f}$ the corresponding local family f_t, such that $f_0 = f$. Since f is hyperbolic and has no critical relations, the following fundamental facts hold: every nearby map f_t is conjugate to f by a quasiconformal homeomorphism h_t, and h_t can be chosen to be analytic in t. Furthermore, the complex dilatations of h_t form a family $\nu(z,t)$ as described above. For $f \in S_d$ and without critical relations, this is an immediate corollary of [29], Theorem 7.4, and [2], Theorem 3. This shows that the vector $\{\sigma'(0), m'(0), v'_1(0), ..., v'_{2d-2}(0)\}$ can be chosen arbitrary, and, hence, proves that Theorem 8 implies Theorem 5.

7 Proof of Proposition 3

We present two proofs of this basic fact. The first proof uses general properties of analytic sets, and it is very similar to the proof of Proposition 1. The second one is a direct and nice construction of the (local) inverse map δ^{-1} with help of quasiconformal surgery. We use an idea by Eremenko and follow essentially [8], where it is done for polynomials and for a single critical value. It gives an alternative proof of Proposition 1 as well.

Both proofs start as follows. Denote $\Lambda = \Lambda_{d,\bar{p}'}$. The map f has a critical point c of multiplicity $m \ge 1$ with a finite critical value $v = f(c)$ if and only if c satisfies the following conditions: $f'(c) = 0, ..., f^{(m)}(c) = 0, \quad f^{(m+1)}(c) \ne 0$. From the latter two conditions, one can express c as a local holomorphic function $c = \phi_m(\bar{f})$ of the vector of the coefficients $\bar{f} \in \mathbf{C}^{2d}$. This determines $m-1$ algebraic equations $\Psi_{k,m}(\bar{f}) = 0$, where $\Psi_{k,m}(\bar{f}) = f^{(k)}(\phi_m(\bar{f}))$, $k = 1, ..., m-1$. If the critical value v is infinite, the conclusion is the same (considering $1/f$), and we will use similar notations in this case as well. Denote $\bar{\Psi}(\bar{f}) = \{\Psi_{k,m_j}(\bar{f}) = 0\}_{j=1,k=1}^{j=p',k=m_j}$. Thus the analytic set Λ in $\mathbf{C}^{2d}$ is determined by the following $2d-2-p'$ equations of the vector $\bar{f} \in \mathbf{C}^{2d}$: $\bar{\Psi}_{k,m_j}(\bar{f}) = 0$.

Secondly, we have the map $\delta : \Lambda \to \mathbf{C}^{p'+2}$ defined by $\delta(\bar{f}) = \bar{v}$. It can be represented as the restriction on Λ of the following map (denoted by $\tilde{\delta}$), which is (locally) holomorphic in $\bar{f} \in \mathbf{C}^{2d}$:

$$\tilde{\delta}(\bar{f}) = \{\sigma(\bar{f}), m(\bar{f}), f(\phi_{m_1}(\bar{f})), ..., f(\phi_{m_p}(\bar{f})), 1/f(\phi_{m_{p+1}}(\bar{f})), ..., 1/f(\phi_{m_{p'}}(\bar{f}))\}.$$

As $\delta : \Lambda \to \mathbf{C}^{p'+2}$ has a holomorphic extension $\tilde{\delta}$, it is enough to prove the following claim: the map $\delta : \Lambda \to \mathbf{C}^{p'+2}$ maps a neighborhood in Λ of every $\bar{f}_0 \in \Lambda$ onto a neighborhood in $\mathbf{C}^{p'+2}$ of the point $\delta(\bar{f}_0)$ and has a local holomorphic inverse δ^{-1}. We present two proofs of this claim.

7.1 First proof

It is similar to the proof of Proposition 1, see Sect. 3.1. The following lemma is crucial:

Lemma 7.1 *The map $\delta : \Lambda \to \mathbf{C}^{p'+2}$ is injective in a neighborhood of every $\bar{f}_0 \in \Lambda$.*

The proof of this Lemma is almost identical to the proof of the injectivity of the map π in Proposition 1, so we omit it.

Next, we use the following well-known statement about analytic sets. Its particular case (for $r = l$) was used to prove Proposition 1.

Proposition 5 *Let U be a ball in $\mathbf{C}^l$, and let E be an analytic set in U, which is defined as the set of common zeros of $l - r$ holomorphic functions in U, for some $0 < r \leq l$. Assume $g : U \to \mathbf{C}^r$ is a holomorphic map, which is injective on E. Then $g(E)$ is an open set in $\mathbf{C}^r$ and $g : E \to \mathbf{C}^r$ has a holomorphic inverse on this set.*

Proposition 3 follows immediately from Lemma 7.1 and Proposition 5, if we set $l = 2d$, $r = p' + 2$, $g = \tilde{\delta}$, U to be a ball around $\bar{f}_0 \in \Lambda$, where δ is injective on Λ, and $E = \Lambda \cap U$.

It remains to prove the above Proposition 5. Consider the restriction $g|_E$ of the holomorphic map $g : U \to \mathbf{C}^r$ on E. Since g is injective on E, every point $z_0 \in E$ is obviously an isolated point in the set $g|_E^{-1}(g(z_0))$. Therefore (see e.g. [38], Chapter 4, Theorem 6B), for some neighborhood W of z_0, the set $F := g(E \cap W)$ is analytic, and the dimension of F at the point $g(z_0)$ is equal to the dimension of E at z_0. On the other hand, the dimension of E at each point is at least r because E is defined in $\mathbf{C}^l$ by $l - r$ equations ([38], Chapter 2, Theorem 12G). Hence, as the analytic set F lies in $\mathbf{C}^r$ and its dimension is at least r, it is equal to r and F is a neighborhood of $g(z_0)$ in $\mathbf{C}^r$. Thus $g(E)$ is open in $\mathbf{C}^r$. Now, the map $g|_E^{-1}$ is well-defined on this open set, and it is analytic in a neighborhood of the image of every regular point of E. On the rest of the points, which form an analytic set of smaller dimension, $g|_E^{-1}$ is locally bounded. By the extended Riemann removable singularity theorem, $g|_E^{-1}$ is holomorphic everywhere.

7.2 Second proof

Let $f_0 \in \Lambda_{d,\bar{p}'}$ and $\bar{v}(f_0) = \delta(\bar{f}_0) = \{\sigma_0, m_0, v_1^0, ..., v_p^0, 0, ..., 0\}$. We prove the existence of a local holomorphic inverse δ^{-1} by constructing a rational function $f \in \Lambda_{d,\bar{p}'}$ with a prescribed $\bar{v} = \bar{v}(f)$ so that $\bar{v}$ is close to $\bar{v}(f_0)$ and $\bar{f}$ depends holomorphically on $\bar{v}$. To this end, choose small (in the spherical metric) pairwise disjoint disks B_k, $k = 1, ..., q_0$, centered at the critical values of f_0. Let D_j, $j = 1, ..., p'$, be the components of f_0-preimages of all B_k on which f_0 is not one-to-one. Each D_j is small (in the Euclidean metric) and contains one and only one critical point c_j^0 of f_0. Given any vector $\bar{v} = \{\sigma, m, v_1, ..., v_p, v_{p+1}^*, ..., v_{p'}^*\}$ close to $\bar{v}(f_0)$, and given $1 \le j \le p'$, one can choose a diffeomorphism ϕ_j of the Riemann sphere, which satisfies the following conditions: (1) $\phi_j(z)$ depends on z and v_j only, and ϕ_j is the identity outside of the ball $B_{k(j)} = f_0(D_j)$, (2) $\phi_j(v_j^0) = v_j$, if $j = 1, ..., p$, and $\phi_j(\infty) = 1/v_j^*$, if $j = p+1, ..., p'$, and (3) ϕ_j depends holomorphically on v_j, if $j = 1, ..., p$, and on v_j^*, if $j = p+1, ..., p'$. Such ϕ_j can be constructed, for example, as in [8]. First, for a disk $B = B(a, r)$, set $\chi_B(z)$ to be 0, if $z \notin B$, and $\chi_B(z) = (1 - |z-a|^2/r^2)^2$, if $z \in B$. Define $\phi_{B,b}(z) = z + (b-a)\chi_B(z)$. If $|b-a|$ is small enough, then $\phi_{B,b}$ is a diffemorphism of $\mathbf{C}$, such that $\phi_{B,b}(a) = b$. For a disk $B = B^*(R)$ around ∞, we denote $B_0 = B(0, 1/R)$ and set $\phi_{B,b} = J \circ \phi_{B_0,1/b} \circ J^{-1}$, where $J(z) = 1/z$. Now we can define: $\phi_j = \phi_{B_{k(j)},v_j}$ for $1 \le j \le p$, and $\phi_j = \phi_{B_\infty,1/v_j^*}$ for $p+1 \le j \le p'$, where B_∞ is the disk centered at ∞, which is among B_k (provided $p < p'$).

Now, define a new function f^*, such that $f^*(z) = f_0(z)$ outside of all D_j and $f^*(z) = \phi_j(f_0(z))$ if $z \in D_j$. Note that $f^* = f_0$ in a definite neighborhood of ∞. Also, $f^*(z)$ depends holomorphically on $\bar{v}$ for every z, and $f^* \to f_0$, as $\bar{v} \to \bar{v}(f_0)$, uniformly on the Riemann sphere. The map f^* is a degree d smooth map of the Riemann sphere with the critical values at $v_1, ..., v_p, 1/v_{p+1}^*, ..., 1/v_{p'}^*$ and with the same expansion at ∞: $f^*(z) = \sigma_0 z + m_0 + O(1/z)$. Let $\mu = \frac{\partial f^*}{\partial \bar{z}} / \frac{\partial f^*}{\partial z}$. As $||\mu||_\infty < 1$, there exists a quasiconformal homeomorphism of the sphere ψ, such that the complex dilatation of ψ is μ, in particular, it is holomoprhic near infinity, and normalized by $\psi(z) = \tilde{a}z + \tilde{b} + O(1/z)$ with any prescribed $\tilde{a} \ne 0, \tilde{b}$. To be more precise, if $\psi^\mu(z)$ is the normalized quasiconformal map as in the beginning of Sect. 3.4, then $\psi^\mu(z) = \rho z + k + O(1/z)$ at ∞, and we define $\psi = a\psi^\mu + b$ with $a = \sigma_0/(\sigma\rho)$ and $b = (m_0 - m)/\sigma - \sigma_0 k/(\sigma\rho)$. For every z, $\psi^\mu(z)$ is holomorphic in $\bar{v}$, and, it follows, that ρ, k depend holomorphically on $\bar{v}$, too. Therefore, $\psi(z)$ is also holomorphic in $\bar{v}$. Finally, define $f = f^* \circ \psi^{-1}$. Then f is rational. Moreover, a, b are chosen so that $f(z) = \sigma z + m + O(1/z)$. It is easy to see that $f(z)$ is a continuous function of $\bar{v}$, and $f = f_0$ for $\bar{v} = \bar{v}(f_0)$. Furthermore, $f(z)$ depends holomorphically on each variable $\sigma, m, v_1, ...$, for every z. One can check this as in [8]: we differentiate the identity $f \circ \psi = f^*$ by $\bar{\partial}\sigma, \bar{\partial}m, \bar{\partial}v_1, ...$ and take into account that ψ, f^* are holomorphic in $\bar{v}$. Thus $f \in \Lambda_{d,\bar{p}'}$, $\bar{v}(f) = \bar{v}$, and $f(z)$ depends on $\bar{v}$ holomorphically for every z. Since $f_0(z) = \sigma_0 z + m_0 + P_0(z)/Q_0(z)$,

where the polynomials P_0 and Q_0 have no common roots, this implies that the vector $\bar{f}$ of the coefficients of f depends holomorphically on $\bar{v}$ as well. It defines a local holomorphic inverse δ^{-1}. By the above, we are done.

8 Proof of Theorem 8

8.1 Beltrami coefficients.

As it has been mentioned already, we derive the theorem with help of quasiconformal deformations. Let $f \in S_d$, and satisfy the conditions (1)-(3) of Sect. 6. Let $\nu(z,t)$ be an analytic family of invariant Beltrami coefficients in the Riemann sphere, such that $\nu(z,0) = 0$. As $\nu(z,t)$ is differentiable at $t = 0$, $\nu(z,t) = t\mu(z) + t\epsilon(z,t)$, where $||\epsilon(z,t)||_\infty \to 0$ as $t \to 0$. We have seen μ is invariant by f, too. Let h_t be an analytic family of quasiconformal homeomorphisms in the plane that fix ∞, so that h_t has the complex dilatation $\nu(z,t)$, and $h_0 = id$. Then $f_t = h_t \circ f \circ h_t^{-1} \in S_d$ is an analytic in t family, and $O_t = h_t(O)$ the corresponding attracting periodic orbit of f_t. Let $\rho(t)$ denote its multiplier, and $v_j(t) = h_t(v_j)$ the critical values of f_t. Define also $\sigma(t)$, $m(t)$ by the expansion $f_t(z) = \sigma(t)z + m(t)z + O(1/z)$ as $z \to \infty$.

8.2 Speed of the multiplier.

Lemma 8.1

$$\frac{\rho'(0)}{\rho} = -\frac{1}{\pi} \lim_{C \to \{b\}} \int_C \frac{\mu(z)}{(z-b)^2} d\sigma_z, \tag{70}$$

where C is a fundamental region near $b \in O$.

$$\frac{\sigma'(0)}{\sigma} = \frac{1}{\pi} \lim_{C_\infty \to \{\infty\}} \int_{C_\infty} \frac{\mu(z)}{z^2} d\sigma_z, \tag{71}$$

where C_∞ is a fundamental region near ∞.

The first equality is the content of Lemma 3.2. The (71) can be obtained from (70) by the change of variable $z \mapsto 1/z$.

8.3 Adjoint identity.

We want to integrate the identity

$$A(z) - (TA)(z) = \sum_{j=1}^{2d-2} \frac{L_j}{z - v_j}. \tag{72}$$

against the f-invariant Beltrami form μ. One cannot do this directly, because A is not integrable at the points of the periodic orbit O as well as at ∞ (if $\mu \neq 0$ near ∞). To deal with this, for every small $r > 0$ and big R, consider the domain $V_{r,R}$ to be the plane $\mathbf{C}$ with the following sets deleted: $f(B^*(R))$ and $B(b_1, r)$ union with $f_{b_{n-k+1}}^{-k}(B(b_1, r))$, for $k = 1, ..., n-1$, where $f_{b_{n-k+1}}^{-k}$ is a local branch of f^{-k} taking $b_1 \in O$ to b_{n-k+1}. In other words,

$$V_{r,R} = \mathbf{C} \setminus \{f(B^*(R)) \cup B(b_1, r) \cup_{k=1}^{n-1} f_{b_{n-k+1}}^{-k}(B(b_1, r))\}.$$

Then A is integrable in $V_{r,R}$, and, therefore,

$$\int_{V_{r,R}} TA(z)\mu(z)d\sigma_z = \int_{f^{-1}(V_{r,R})} A(z)\mu(z).$$

Now, $f^{-1}(V_{r,R}) = V_{r,R} \setminus (C_r \cup C_R^* \cup \Delta_r \cup \Delta_R^*)$, where $C_r = f_{b_1}^{-n}(B(b_1, r)) \setminus B(b_1, r)$ is a fundamental region near b_1, and $C_R^* = B^*(R) \setminus f(B^*(R))$ is a fundamental region near infinity (defined by the local branches $f_{b_1}^{-n}$ that fixes b_1 and f_∞^{-1} that fixes ∞ resp.), and, in turn, Δ_r and Δ_R^* are open set which are away from O and ∞, and which shrink to a finitely many points as $r \to 0$ and $R \to \infty$ resp. Therefore,

$$\int_{V_{r,R}} (A(z) - TA(z))\mu(z)d\sigma_z = \int_{C_r} A(z)\mu(z)d\sigma_z + \int_{C_R^*} A(z)\mu(z)d\sigma_z + o_r(1) + o_R^\infty(1). \tag{73}$$

Here and below little-o notation mean that $o_r(1) \to 0$ as $r \to 0$ and $o_R^\infty(1) \to 0$ as $R \to \infty$. It is easy to see that

$$\int_{C_r} A(z)\mu(z)d\sigma_z = \int_{C_r} \frac{\mu(z)}{(z-b_1)^2} d\sigma_z + o_r(1)$$

and

$$\int_{C_R^*} A(z)\mu(z)d\sigma_z = \Gamma_1 \int_{C_R^*} \frac{\mu(z)}{z} + \Gamma_2 \int_{C_R^*} \frac{\mu(z)}{z^2} + o_R^\infty(1),$$

where Γ_1, Γ_2 are defined by the expansion $A(z) = \Gamma_1/z + \Gamma_2/z^2 + O(1/z^3)$ at infinity. Thus,

$$\int_{V_{r,R}} (A(z) - TA(z))\mu(z)d\sigma_z = \int_{C_r} \frac{\mu(z)}{(z-b_1)^2} d\sigma_z + \Gamma_1 \int_{C_R^*} \frac{\mu(z)}{z} d\sigma_z + \Gamma_2 \int_{C_R^*} \frac{\mu(z)}{z^2} d\sigma_z + o_r(1) + o_R^\infty(1). \tag{74}$$

The identity (72) then gives us:

$$\int_{C_r} \frac{\mu(z)}{(z-b_1)^2} d\sigma_z + \Gamma_1 \int_{C_R^*} \frac{\mu(z)}{z} d\sigma_z + \Gamma_2 \int_{C_R^*} \frac{\mu(z)}{z^2} d\sigma_z + o_r(1) + o_R^\infty(1) = \sum_{j=1}^{2d-2} L_j \int_{V_{r,R}} \frac{\mu(z)}{z - v_j} d\sigma_z \tag{75}$$

Lemma 8.1 allows us to pass to the limit as $r \to 0$:

$$- \pi \frac{\rho'(0)}{\rho} + \Gamma_1 \int_{C_R^*} \frac{\mu(z)}{z} d\sigma_z + \Gamma_2 \int_{C_R^*} \frac{\mu(z)}{z^2} d\sigma_z + o_R^\infty(1) = \sum_{j=1}^{2d-2} L_j \int_{V_R} \frac{\mu(z)}{z - v_j} d\sigma_z, \tag{76}$$

where

$$V_R = \mathbf{C} \setminus f(B^*(R)).$$

By the same Lemma 8.1, in the equation (76) one can write the asymptotics as $R \to \infty$. We get:

$$- \pi \frac{\rho'(0)}{\rho} + \Gamma_2 \pi \frac{\sigma'(0)}{\sigma} + \Gamma_1 \int_{C_R^*} \frac{\mu(z)}{z} d\sigma_z + o_R^\infty(1) = \sum_{j=1}^{2d-2} L_j \int_{V_R} \frac{\mu(z)}{z - v_j} d\sigma_z. \tag{77}$$

Speed of critical values. Now we want to express the integral of $\mu(z)/(z-v_j)$ via $v_j'(0)$. The difference with the polynomial case is that μ does not vanish at infinity anymore. Let ψ_t be the quasiconformal homeomorphism of the plane with the complex dilatation $\nu(z,t)$, that fixes $0, 1$ and ∞. As h_t has the same complex dilatation and fixes ∞ too, we have: $h_t = a(t)\psi_t + b(t)$, where a, b are analytic in t, and $a(0) = 1$, $b(0) = 0$. Using (40)-(42), we can write

$$v_j'(0) = a'(0) v_j + b'(0) - \frac{1}{\pi} \int_{V_R} \frac{\mu(z)}{z - v_j} d\sigma_z - \frac{1}{\pi} \int_{V_R} \mu(z) \left(\frac{v_j - 1}{z} - \frac{v_j}{z-1}\right) d\sigma_z + o_R^\infty(1). \tag{78}$$

From this and (76), we obtain, then,

$$\frac{\rho'(0)}{\rho} = \Gamma_2 \frac{\sigma'(0)}{\sigma} + \sum_{j=1}^{2d-2} L_j v_j'(0) + \Delta, \tag{79}$$

where

$$\Delta = -b'(0) \sum_{j=1}^{2d-2} L_j - a'(0) \sum_{j=1}^{2d-2} v_j L_j +$$

$$\lim_{R\to\infty} \left\{ \frac{\Gamma_1}{\pi} \int_{C_R^*} \frac{\mu(z)}{z} d\sigma_z - \frac{1}{\pi} \int_{V_R} \frac{\mu(z)}{z} d\sigma_z \sum_{j=1}^{2d-2} L_j - \frac{1}{\pi} \int_{V_R} \frac{\mu(z)}{z(z-1)} d\sigma_z \sum_{j=1}^{2d-2} v_j L_j \right\}.$$

Let us find connections between Γ_1, Γ_2, and L_j, v_j:

Lemma 8.2

$$\frac{\sigma - 1}{\sigma} \Gamma_1 = \sum_{j=1}^{2d-2} L_j, \quad \frac{m}{\sigma} \Gamma_1 = - \sum_{j=1}^{2d-2} v_j L_j. \tag{80}$$

Proof. By definition, $A(z) = \Gamma_1/z + \Gamma_2/z^2 + O(1/z^3)$. If a point e of the plane is such that $f(e) = \infty$, and w is close to e, so that $z = f(w)$ is close to ∞, then it is easy to check that $f'(w) \sim Cz^2$ as $z \to \infty$, where $C \neq 0$. We see that the asymptotics of $TA(z)$ at ∞ up to $1/z^2$ is defined by the preimage w_∞ of z, which is close to ∞. In turn, $w_\infty = (z-m)/\sigma + O(1/z)$. It gives us:

$$TA(z) = \frac{\Gamma_1}{\sigma}\frac{1}{z} + (\Gamma_2 + \frac{m\Gamma_1}{\sigma})\frac{1}{z^2} + O(\frac{1}{z^3}). \tag{81}$$

Therefore,

$$A(z) - TA(z) = \frac{\sigma - 1}{\sigma}\Gamma_1\frac{1}{z} - \frac{m\Gamma_1}{\sigma}\frac{1}{z^2} + O(\frac{1}{z^3}). \tag{82}$$

Comparing the latter asymptotics with the asymptotics at ∞ of $\sum_j \frac{L_j}{z-v_j}$, we get the statement.

□

Note that (81)-(82) will be used also later on in the proof of Theorem 6.

Let us continue. In view of the latter connections, we can write that

$$\Delta = \frac{\Gamma_1}{\sigma}\Delta_0, \tag{83}$$

and

$$\Delta_0 = ma'(0) - (\sigma-1)b'(0) + \frac{1}{\pi}\lim_{R\to\infty}\{\sigma\int_{C_R^*}\frac{\mu(z)}{z}d\sigma_z - (\sigma-1)\int_{V_R}\frac{\mu(z)}{z}d\sigma_z + m\int_{V_R}\frac{\mu(z)}{z(z-1)}d\sigma_z\}, \tag{84}$$

where

$$V_R = \mathbf{C}\setminus f(B^*(R)), \quad C_R^* = B^*(R)\setminus f(B^*(R)).$$

Our aim is to show that

$$\Delta_0 = m'(0).$$

Evaluation of $m'(0)$. Here we solve the following general problem: calculate $m'(0)$, where $f_t = h_t \circ f \circ h_t^{-1}$ is the quasiconformal deformation of f, such that $f_t(z) = \sigma(t)z + m(t) + O(1/z)$ at ∞. (Note that $\sigma'(0)$ has been calculated in Lemma 8.1.) We get from $f_t \circ h_t = h_t \circ f$, that $f_t(z) = f(z) + tV(z) + O(t^2)$ with

$$V(z) = a'(0)(f(z) - zf'(z)) + b'(0)(1 - f'(z)) + \kappa(f(z)) - f'(z)\kappa(z), \tag{85}$$

and

$$\kappa(z) = \frac{\partial\psi_t}{\partial t}|_{t=0}(z) = -\frac{1}{\pi}\int_{\mathbf{C}}\mu(w)\frac{z(z-1)}{w(w-1)(w-z)}d\sigma_w. \tag{86}$$

Using the asymptotics of f at ∞, we proceed:

$$V(z) = ma'(0) + (1-\sigma)b'(0) + \kappa(f(z)) - f'(z)\kappa(z) + O(\frac{1}{z}). \tag{87}$$

Note that V is a rational function of z. In particular, it is meromorphic at ∞. Therefore,

$$m'(0) = \frac{1}{2\pi i}\int_{|z|=R} \frac{V(z)}{z}dz = ma'(0)+(1-\sigma)b'(0)+\frac{1}{2\pi i}\int_{|z|=R} \frac{\kappa(f(z)) - f'(z)\kappa(z)}{z}dz, \tag{88}$$

for every R large enough. Now we need to calculate

$$J = \frac{1}{2\pi i}\int_{|z|=R} \frac{\kappa(f(z)) - f'(z)\kappa(z)}{z}dz. \tag{89}$$

By (86) and Fubini's theorem:

$$J = -\frac{1}{\pi}\int_{\mathbf{C}} \frac{\mu(w)}{w(w-1)}d\sigma_w \frac{1}{2\pi i}\int_{|z|=R} \frac{1}{z}[\frac{f(z)(f(z)-1)}{w-f(z)} - \frac{f'(z)z(z-1)}{w-z}]dz. \tag{90}$$

Denote the internal integral

$$I(w,R) = \frac{1}{2\pi i}\int_{|z|=R} F(z,w)dz, \tag{91}$$

where

$$F(z,w) = \frac{1}{z}[\frac{f(z)(f(z)-1)}{w-f(z)} - \frac{f'(z)z(z-1)}{w-z}]. \tag{92}$$

For a large (but fixed!) R, we calculate $I(w,R)$ for different w using the Residue Theorem. First, it is easy to check that, as $z \to \infty$,

$$F(z,w) = \frac{1}{z}\frac{w[f(z)(f(z)-1) - f'(z)z(z-1)] + [f'(z)f(z)z(z-1) - zf(z)(f(z)-1)]}{(w-f(z))(w-z)} =$$

$$= \frac{(\sigma-1)(w-1)-m}{z} + O(\frac{1}{z^2}).$$

Therefore,

$$I(w,R) = [(\sigma-1)(w-1)-m] - \sum_{|z|>R} ResF(z,w). \tag{93}$$

The result depends on the position of w.

(i) If $|w| < R$, then $F(z,w)$ has no singular points for $|z| \geq R$. Therefore,

$$I(w,R) = (\sigma-1)(w-1)-m. \tag{94}$$

(ii) If $|w| > R$ and $w \in \mathbf{C} \setminus f(B^*(R))$, then $F(z,w)$ has a single singular point in $|z| > R$ at $z = w$. Therefore,

$$I(w,R) = [(\sigma-1)(w-1) - m] - f'(w)(w-1) = -(w-1) - m + O(\frac{1}{w}). \quad (95)$$

(iii) If $w \in f(B^*(R))$, then $F(z,w)$ has two singular points in $|z| > R$: at $z = w$ and at a unique $z = z_w$ in this domain, such that $f(z_w) = w$. Therefore,

$$I(w,R) = [(\sigma-1)(w-1) - m] - f'(w)(w-1) + \frac{w(w-1)}{z_w f'(z_w)}. \quad (96)$$

We have: $z_w = (w-m)/\sigma + O(1/w)$ so, hence, after some straightforward manipulations,

$$I(w,R) = [(\sigma-1)(w-1) - m] - f'(w)(w-1) + \frac{w(w-1)}{z_w f'(z_w)} = O(\frac{1}{w}). \quad (97)$$

With help of (i)-(iii), we calculate

$$J = -\frac{1}{\pi}\int_{\mathbf{C}} \frac{\mu(w)}{w(w-1)} I(w,R) d\sigma_w = J_1 + J_2 + J_3 \quad (98)$$

as follows.

$$J_1 = -\frac{1}{\pi}\int_{|w|<R} \frac{\mu(w)}{w(w-1)}[\sigma-1)(w-1) - m] d\sigma_w =$$

$$-\frac{1}{\pi}(\sigma-1)\int_{|w|<R} \frac{\mu(w)}{w} d\sigma_w + \frac{1}{\pi} m \int_{|w|<R} \frac{\mu(w)}{w(w-1)} d\sigma_w,$$

and

$$J_2 = -\frac{1}{\pi}\int_{B^*(R)\setminus f(B^*(R))} \frac{\mu(w)}{w(w-1)}[-(w-1) - m + O(\frac{1}{w})] d\sigma_w =$$

$$\frac{1}{\pi}\int_{B^*(R)\setminus f(B^*(R))} \frac{\mu(w)}{w} d\sigma_w + \frac{m}{\pi}\int_{B^*(R)\setminus f(B^*(R))} \frac{\mu(w)}{w(w-1)} d\sigma_w + \int_{B^*(R)\setminus f(B^*(R))} O(\frac{1}{w^3}) d\sigma_w,$$

and, at last,

$$J_3 = -\frac{1}{\pi}\int_{f(B^*(R))} \frac{\mu(w)}{w(w-1)} O(\frac{1}{w}) d\sigma_w = \int_{f(B^*(R))} O(\frac{1}{w^3}) d\sigma_w.$$

Since J is independent on R, we can write then:

$$m'(0) = ma'(0) + (1-\sigma)b'(0) - \frac{1}{\pi}\lim_{R\to\infty}\{(\sigma-1)\int_{|w|<R} \frac{\mu(w)}{w} d\sigma_w - \int_{B^*(R)\setminus f(B^*(R)} \frac{\mu(w)}{w} d\sigma_w -$$

$$m\int_{|w|<R} \frac{\mu(w)}{w(w-1)} d\sigma_w - m\int_{B^*(R)\setminus f(B^*(R))} \frac{\mu(w)}{w(w-1)} d\sigma_w\}.$$

In other words, we have proved:

Lemma 8.3

$$m'(0) = ma'(0) + (1-\sigma)b'(0) +$$
$$\frac{1}{\pi}\lim_{R\to\infty}\{-(\sigma-1)\int_{\mathbf{C}\setminus f(B^*(R))}\frac{\mu(w)}{w}d\sigma_w + m\int_{\mathbf{C}\setminus f(B^*(R))}\frac{\mu(w)}{w(w-1)}d\sigma_w + \sigma\int_{B^*(R)\setminus f(B^*(R))}\frac{\mu(w)}{w}d\sigma_w\}.$$

If we compare the latter expression for $m'(0)$ to the expression (84) for Δ_0, we see that $\Delta_0 = m'(0)$. This finishes the proof of Theorem 8, and, therefore, Theorem 5, for $f \in S_d$.

9 More generality: multiple critical points yet finite critical values

Let $f \in \Lambda_{d,\bar{p}}$, and assume that all critical values of f are finite: if c_j are all different critical points of f, then $v_j = f(c_j) \neq \infty$, $1 \le j \le p$. As usual, m_j is the multiplicity of c_j. By Theorem 3, we have:

$$B(z) - (TB)(z) = \sum_{j=1}^{p} \frac{\tilde{L}_j}{z - v_j}, \tag{99}$$

where

$$\tilde{L}_j = -\frac{1}{(m_j-1)!}\frac{d^{m_j-1}}{dw^{m_j-1}}|_{w=c_j}(\frac{B(w)}{Q_j(w)}), \tag{100}$$

where, in turn, Q_j is a local analytic function near c_j defined by $f'(z) = (z - c_j)^{m_j} Q_j(z)$, so that $Q_j(c_j) \neq 0$. What we need to check is that

$$\tilde{L}_j = \frac{\partial \rho}{\partial v_j}. \tag{101}$$

Without loss of generality, $j = 1$. The idea is as follows. Using the coordinates $\bar{v}(f)$, we consider a small analytic path f_t through f in the space $\Lambda_{d,\bar{p}}$, such that only v_1 changes along this path. Then we perturb each f_t in such a way, that all critical points of the perturbed map are simple, and apply Theorem 5 in the proven case of simple critical points. Then we get an integral formula for the variation of ρ along the path, which will imply (101).

Let us do the required analytic work. By Proposition 3, there exists a family $\{f_t\}$ of maps from $\Lambda_{d,\bar{p}}$ of a complex parameter t, $|t| < \delta$, where $\delta > 0$ is small, such that $f_0 = f$ and, for every t, $\bar{v}(f_t) = \{\sigma, m, v_1 + t\delta, v_2, ..., v_p\}$. Denote by $c_j(t)$ a critical point of f_t, the continuation of the critical point c_j. If δ is small enough, the periodic orbit O of f extends holomorphically to a periodic orbit O_t of f_t. We perturb f_t as follows: given ϵ with small enough modulus, define $f_{t,\epsilon}(z) = f_t(z) + \epsilon z$.

It is easy to see, that all critical points of $f_{t,\epsilon}$ are simple, that is, $f_{t,\epsilon} \in S_d$. To be more precise, for every $|\epsilon| \neq 0$ small enough, to every critical point $c_j(t)$ of f_t there corresponds m_j simple critical points $c_{j,k}(t,\epsilon)$, $k = 1, ..., m_j$, of $f_{t,\epsilon}$, so that $\lim_{\epsilon \to 0} c_{j,k}(t,\epsilon) = c_j(t)$. Denote $v_{j,k}(t,\epsilon) = f_{t,\epsilon}(c_{j,k}(t,\epsilon))$. Furthermore, if δ, $|\epsilon|$ are small, the periodic points of O extends to holomorphic functions of t, ϵ and form a periodic orbit $O(t,\epsilon)$ of $f_{t,\epsilon}$. Its multiplier $\hat{\rho}(t,\epsilon)$ is a holomorphic function in t, ϵ, too. In particular, $\hat{\rho}(t,0) = \hat{\rho}(t)$, the multiplier of O_t for f_t. We have:

$$\hat{\rho}(t) - \hat{\rho}(0) = \lim_{\epsilon \to 0} \{\hat{\rho}(t,\epsilon) - \hat{\rho}(0,\epsilon)\}.$$

We fix $\epsilon \neq 0$ and calculate $\hat{\rho}(t,\epsilon) - \hat{\rho}(0,\epsilon)$ as follows. Since all critical points of $f_{t,\epsilon}$ are simple, then $\hat{\rho}(t,\epsilon)$ is a holomorphic function $\rho(\bar{v}(f_{t,\epsilon}))$ of $\bar{v}(f_{t,\epsilon}) = \{\sigma(t,\epsilon), m(t,\epsilon), \{v_{j,k}(t,\epsilon)\}_{j=1,...,p;k=1,...,m_j}\}$. Now, $f_{t,\epsilon}(z) = (\sigma+\epsilon)z + m + O(1/z)$, hence, $\sigma(t,\epsilon) = \sigma + \epsilon, m(t,\epsilon) = m$.

We will denote by $(z)_t = \partial z/\partial t$. In particular, $(\sigma)_t(t,\epsilon) = (m)_t(t,\epsilon) = 0$. We have:

$$\hat{\rho}(t_0,\epsilon) - \hat{\rho}(0,\epsilon) = \int_0^{t_0} (\hat{\rho})_t(t,\epsilon) dt =$$

$$\int_0^{t_0} (\frac{\partial \rho}{\partial \sigma}(\sigma)_t(t,\epsilon) + \frac{\partial \rho}{\partial m}(m)_t(t,\epsilon) + \sum_{j=1}^{p}\sum_{k=1}^{m_j} \frac{\partial \rho}{\partial v_{j,k}}(v_{j,k})_t(t,\epsilon))dt = \int_0^{t_0} (\sum_{j=1}^{p}\sum_{k=1}^{m_j} \frac{\partial \rho}{\partial v_{j,k}}(v_{j,k})_t(t,\epsilon))dt.$$

By the proven formulae for $f_{t,\epsilon} \in S_d$,

$$\frac{\partial \rho}{\partial v_{j,k}} = -\frac{B_{O(t,\epsilon)}(c_{j,k}(t,\epsilon))}{f_{t,\epsilon}''(c_{j,k}(t,\epsilon))}.$$

Also,

$$(v_{j,k})_t(t,\epsilon) = \frac{d}{dt} f_{t,\epsilon}(c_{j,k}(t,\epsilon)) = (f_{t,\epsilon})_t(c_{j,k}(t,\epsilon)) + f'_{t,\epsilon}(c_{j,k}(t,\epsilon))(c_{j,k})_t(t,\epsilon) = (f_{t,\epsilon})_t(c_{j,k}(t,\epsilon)).$$

Taking this into account, we can find $r > 0$ fixed and write:

$$\hat{\rho}(t_0,\epsilon) - \hat{\rho}(0,\epsilon) = -\int_0^{t_0} (\sum_{j=1}^{p}\sum_{k=1}^{m_j} \frac{B_{O(t,\epsilon)}(c_{j,k}(t,\epsilon))}{f_{t,\epsilon}''(c_{j,k}(t,\epsilon))}(f_{t,\epsilon})_t(c_{j,k}(t,\epsilon)))dt =$$

$$-\int_0^{t_0} (\sum_{j=1}^{p} \frac{1}{2\pi i} \int_{|w-c_j(t)|=r} \frac{B_{O(t,\epsilon)}(w)}{f'_{t,\epsilon}(w)}(f_{t,\epsilon})_t(w)dw)dt.$$

Passing to a limit as $\epsilon \to 0$, we obtain:

$$\hat{\rho}(t_0) - \hat{\rho}(0) = -\int_0^{t_0} (\sum_{j=1}^{p} \frac{1}{2\pi i} \int_{|w-c_j(t)|=r} \frac{B_{O_t}(w)}{f'_t(w)}(f_t)_t(w)dw)dt.$$

It follows from $f_t(w) = v_j(t) + O(w - c_j(t))^{m_j+1}$, that

$$(f_t)_t(w) = (v_j)_t(t) + O(w - c_j(t))^{m_j}.$$

On the other hand, by the choice of f_t, $(v_1)_t(t) = 1$ while $(v_j)_t(t) = 0$ for $2 \le j \le p$. Therefore,

$$\hat{\rho}(t_0) - \hat{\rho}(0) = \int_0^{t_0} L_1(t)dt,$$

where

$$L_1(t) = -\frac{1}{2\pi i}\int_{|w-c_1(t)|=r} \frac{B_{O_t}(w)}{f_t'(w)}dw.$$

In the local coordinates of $\Lambda_{d,\bar{p}}$ near $f_0 = f$,

$$\rho(\sigma, m, v_1 + t_0, v_2, ..., v_p) - \rho(\sigma, m, v_1, ..., v_p) = \int_0^{t_0} L_1(t)dt.$$

It yields immediately, that

$$\frac{\partial \rho}{\partial v_1} = -\frac{1}{2\pi i}\int_{|w-c_1|=r} \frac{B(w)}{f'(w)}dw = L_1(0).$$

10 Finishing the proof of Theorem 5: the case of infinite critical values

Let $f_0 \in \Lambda_{d,\bar{p}'}$ have some critical values equal to ∞, and O be a periodic orbit of f_0 with the multiplier different from 1. Without loss of generality one can assume that f_0 has different critical points $c_1^0, ..., c_{q'}^0$, so that the critical values $v_j^0 = f_0(c_j^0)$ are finite for $1 \le j \le p < p'$, and $v_j^0 = \infty$ for $p < j \le p'$. As usual, m_j denotes the multiplicity of c_j^0. By Proposition 3, $\bar{v}(f) = (\sigma, m, v_1, ..., v_p, 1/v_{p+1}, ..., 1/v_{p'})$ are local coordinates in $\Lambda_{d,\bar{p}'}$ in a neighborhood of $\bar{v}(f_0) = (\sigma_0, m_0, v_1^0, ..., v_p^0, 0, ..., 0) \in \mathbf{C}^{p'+2}$. Therefore, f and ρ are holomorphic in $\bar{v}(f)$ in a neighborhood of $\bar{v}(f_0)$. In particular, for every $1 \le j \le p'$, $\partial\rho/\partial v_j$ is continuous in $\bar{v}(f)$ at $\bar{v}(f_0)$. Consider now a sequence $f_n \in \Lambda_{d,\bar{p}'}$, such that all critical values of each f_n are finite, and $\bar{v}(f_n) \to \bar{v}(f_0)$. Let O_n be the periodic orbit of f_n, such that $O_n \to O$ as $n \to \infty$, ρ_n the multiplier of O_n, and let B_n and T_n be the associated rational function for O_n and the transfer operator for f_n. Clearly, $B_n \to B$, $T_n \to T$, the corresponding objects for f_0. To prove Theorem 5 for f_0, it remains to show two facts:

(a) $\partial\rho_n/\partial v_j \to 0$, $n \to \infty$, for $p < j \le p'$,

(b)

$$\frac{\partial \rho_n}{\partial(1/v_j)} = -v_j(f_n)^2\frac{\partial \rho_n}{\partial v_j} \to \frac{1}{2\pi i}\int_{|w-c_j^0|=r} \frac{B(w)}{(1/f_0)'(w)}dw$$

$$= \frac{1}{(m_j-1)!}\frac{d^{m_j}}{dw^{m_j}}|_{w=c_j^0}(\frac{B}{Q_j}),$$

$n \to \infty$, for $p < j \leq p'$, where $(1/f_0)'(w) = (w - c_j^0)^{m_j} Q_j(w)$.

Since $v_j(f_n) \to \infty$, $p < j \leq p'$, then (a) follows from (b). To prove (b), let us fix $p < j \leq p'$. Then fix n, and denote $c = c_j(f_n)$, $m = m_j$, and Q, such that $f_n'(w) = (w-c)^m Q(w)$. As $w \to c$, we can write

$$\frac{(f_n(w))^2 B_n(w)}{(f_n'(w))^2} = \frac{(f_n(c) + O(w-c)^{m+1})^2 B_n(w)}{(w-c)^m (Q(w))^2} = \frac{(f_n(c))^2}{(w-c)^m (Q(w))^2} + O(w-c).$$

Therefore, for $r_n > 0$ small enough and every $p < j \leq p'$,

$$(v_j(f_n))^2 \frac{\partial \rho_n}{\partial v_j} = -\frac{1}{2\pi i}\int_{|w-c_j(f_n)|=r_n} \frac{(f_n(w))^2 B_n(w)}{f_n'(w)} dw.$$

For all n large, one can increase r_n to a size, which is independent on n. Indeed, $f_n \to f_0$ in $\Lambda_{d,\bar{p}'}$. Hence, for a point w_0 near $c_j(f_n)$, if $f_n(w_0) = \infty$, then $f_n \sim C_n/(w - w_0)$, $w \to w_0$, and $(f_n(w))^2/f_n'(w)$ has no singularity at w_0. This shows that one can fix $r_n = r > 0$. Then, for every $p < j \leq p'$ and fixed r, we can pass to a limit as $n \to \infty$:

$$-v_j(f_n)^2 \frac{\partial \rho_n}{\partial v_j} = \frac{1}{2\pi i}\int_{|w-c_j^0|=r} \frac{(f_n(w))^2 B_n(w)}{f_n'(w)} dw \to \frac{1}{2\pi i}\int_{|w-c_j^0|=r} \frac{(f_0(w))^2 B(w)}{f_0'(w)} dw.$$

The proof of Theorem 5 is completed.

11 Multipliers as local coordinates

11.1 Contraction of the operator T

We need the following Proposition 6. A more general statement, with a careful consideration of parabolic points, is contained in [9].

Proposition 6 *Let P be a non-empty union of some non-repelling periodic orbits of $f \in \Lambda_{d,\bar{p}}$. Consider a non-zero rational function ψ, such that:*

(i) as $z \to \infty$, one of the following conditions hold: either (a) $\psi(z) = O(1/z^3)$, or (b) $\psi(z) = O(1/z)$ and $f(z) = \sigma z + O(1/z)$, where $|\sigma| \geq 1$, or (c) $\psi(z) = O(1/z^2)$ and $f(z) = z + m + O(1/z)$,

(ii) if $b \in P$ is a point of a periodic orbit O, then ψ has either double pole at every point of O, or at most simple pole at every point of O; moreover, ψ has at most simple poles outside of the set P.

Then ψ is not a fixed point of the operator T.

Proof. (cf. [9], [6], see also [13], [14]) Denote $\hat{P} = \cup_j O_j$, where O_j are different periodic orbits from the set P, such that ψ has double pole at every point of $O_j = \{b_k^j\}_{k=1}^{n_j}$. (If ψ has no double poles, then $\hat{P} = \emptyset$.) Denote by ρ_j the multiplier of O_j. Given $r > 0$ small enough, we define a domain $V_r = \mathbf{C} \setminus \{W_\infty \cup \cup_j W_j\}$, where:

(i) if $\psi(z) = O(1/z^3)$ at infinity, then W_∞ is empty, and otherwise define $W_\infty = B^*(1/r)$, a neighborhood of ∞.

(ii) for every O_j, the set W_j is the disk $B(b_1^j, r)$ union with $f_j^{-k}(B(b_1^j, r))$, for $1 \le k \le n_j - 1$, where f_j^{-k} is a local branch of f^{-k} taking $b_1^j \in O_j$ to $b_{n_j-k+1}^j$.

Obviously, $V_r \to \mathbf{C}$ as $r \to 0$, and ψ is integrable in V_r, for $r > 0$. Let j', j", and j''' denote the indexes corresponding to attracting, neutral, and superattracting periodic orbits in $\hat{P}$ respectively. For every j, fix a positive number a_j, such that $|f^{n_j}(z) - b_1^j - \rho_j(z - b_1^j)| < a_j|z - b_1^j|^2$, for $|z - b_1^j| < r$, r small enough. Also, for $f(z) = \sigma z + m + O(1/z)$, fix $a > 0$, such that $|f(z) - \sigma z - m| < a/|z|$, for $|z|$ large. Now we have:

$$f^{-1}(V_r) \subset V_r \cup \hat{W}_\infty \cup \cup_{j"} \hat{W}_{j"} \setminus \{\cup_{j'} \hat{W}_{j'} \cup \cup_{j'''} \hat{W}_{j'''}\}.$$

Here:

(1) $\hat{W}_\infty$ is empty if either W_∞ is empty or $|\sigma| > 1$, and $\hat{W}_\infty \subset B^*(1/r) \setminus B^*(1/r + |m| + ar)$ otherwise (here $m = 0$ in the case (i)(b)).

(2) $\hat{W}_{j"} \subset B(b_1^{j"}, r) \setminus B(b_1^{j"}, r - a_{j"} r^2)$, for every neutral periodic orbit $O_{j"}$.

(3) $B(b_1^{j'}, |\rho_{j'}|^{-1} r - a_{j'} r^2) \setminus B(b_1^{j'}, r) \subset \hat{W}_{j'}$, for every attracting though not superattracting periodic orbit $O_{j'}$.

(3') $B(b_1^{j'''}, 2r) \setminus B(b_1^{j'''}, r) \subset \hat{W}_{j'''}$, if $O_{j'''}$ is superattracting.

It follows (see also the proof of Theorem 2) that:

$$\lim_{r \to 0} \int_{\hat{W}_\infty} |\psi| d\sigma = 0, \quad \lim_{r \to 0} \int_{\hat{W}_{j"}} |\psi| d\sigma = 0, \tag{102}$$

$$\liminf_{r \to 0} \int_{\hat{W}_{j'}} |\psi| d\sigma \ge 2\pi |A_{j'}| \log |\rho_{j'}|^{-1}, \quad \liminf_{r \to 0} \int_{\hat{W}_{j'''}} |\psi| d\sigma \ge 2\pi |A_{j'''}| \log 2, \tag{103}$$

where $A_j \neq 0$ is so that $\psi(z) \sim A_j/(z - b_1^j)^2$. Therefore, under the conditions (i)-(ii),

$$\limsup_{r \to 0} \{ \int_{f^{-1}(V_r)} |\psi(z)| d\sigma_z - \int_{V_r} |\psi(z)| d\sigma_z \} \le 0, \tag{104}$$

and, moreover, the inequality is strict, if ψ has at least one double pole at an attracting or superattracting point of P. Assume now the contrary, i.e. $\psi = T\psi$. If the inequality in (104) is indeed strict, we get at once a contradiction as in the proof of Theorem 2: $0 = \int_{V_r} |\psi| - \int_{V_r} |T\psi| \ge \int_{V_r} |\psi| - \int_{f^{-1}(V_r)} |\psi| > 0$, for

some $r > 0$. This proves that ψ cannot have double poles at attracting and superattracting points of P. Let us show further that

$$|\sum_{w:f(w)=z} \frac{\psi(w)}{(f'(w))^2}| = \sum_{w:f(w)=z} \frac{|\psi(w)|}{|f'(w)|^2}, \tag{105}$$

for every $z \in \mathbf{C}$, where both sides are finite. Indeed, assume that, for some z_0, there is the strict inequality in (105). Then there are a neighborhood U of z_0 and $\epsilon > 0$, such that, for $z \in U$,

$$|\sum_{w:f(w)=z} \frac{\psi(w)}{(f'(w))^2}| < (1-\epsilon) \sum_{w:f(w)=z} \frac{|\psi(w)|}{|f'(w)|^2}. \tag{106}$$

One writes, for $r > 0$ small enough (so that $U \subset V_r$):

$$\int_{V_r} |\psi(z)| d\sigma_z = \int_{V_r} |(T\psi)(z)| d\sigma_z = \int_{V_r \setminus U} |(T\psi)(z)| d\sigma_z + \int_U |(T\psi)(z)| d\sigma_z <$$

$$\int_{f^{-1}(V_r \setminus U)} |\psi(z)| d\sigma_z + (1-\epsilon) \int_{f^{-1}(U)} |\psi(z)| d\sigma_z = \int_{f^{-1}(V_r)} |\psi(z)| d\sigma_z - \epsilon \int_{f^{-1}(U)} |\psi(z)| d\sigma_z.$$

Taking into account (104), we conclude that $\psi = 0$ on U, hence, everywhere. This contradiction shows that (105) holds for every z as above. In turn, (105) and $T\psi = \psi$ imply, that a meromorphic function $\psi \circ f (f')^2/\psi$ takes only positive values, hence, this function is the constant $d = \deg f$. Now, consider the identity $\psi(f^n(z))((f^n)'(z))^2 = d^n \psi(z)$ near a point $b \in P$ of period n and with multiplier ρ, and plug in it the local expansion for f^n and $\psi(z) \sim A(z-b)^l$, with $A \neq 0$ and $l \geq -2$, near b. Then we see that: if $\rho \neq 0$, then $\rho^{l+2} = d^n$, i.e. $l > -2$ and $|\rho| > 1$; if $\rho = 0$, then $l = -2$. In either case, it is a contradiction.

□

11.2 Proof of Theorem 6

Consider first the case (H_∞). Assume the contrary: the rank of the matrix

$$\mathbf{O} = (\frac{\partial \rho_j}{\partial \sigma}, \frac{\tilde{\partial}^V \rho_j}{\partial V_1}, ..., \frac{\tilde{\partial}^V \rho_j}{\partial V_{k-1}}, \frac{\tilde{\partial}^V \rho_j}{\partial V_{k+1}}, \frac{\tilde{\partial}^V \rho_j}{\partial V_q})_{1 \leq j \leq r} \tag{107}$$

is less than r. Without loss of generality, one can assume that $k = q$. Then the vectors $(\frac{\partial \rho_j}{\partial \sigma}, \frac{\tilde{\partial}^V \rho_j}{\partial V_1}, ..., \frac{\tilde{\partial}^V \rho_j}{\partial V_{q-1}})$, $1 \leq j \leq r$, are linearly dependent.

Let $O_j = \{b_k^j\}_{k=1}^{n_j}$, the set of points of the periodic orbit O_j of period n_j, the function $\tilde{B}_j$ is said to be B_{O_j} iff $\rho_j \neq 1$ and $\hat{B}_{O_j}$ iff $\rho_j = 1$. Precisely like

in the proof of Theorem 2, Sect. 4, we see that each $\tilde{B}_j$ is not identically zero. The connections (56), (63) read: $\tilde{B}_j(z) - (T\tilde{B}_j)(z) = \sum_{i=1}^{q} \frac{\tilde{\partial}^V \rho_j}{\partial V_i} \frac{1}{z-V_i}$, for every $j = 1, ..., r$. By the assumption, there exists a linear combination $\psi = \sum_{j=1}^{r} \beta_j \tilde{B}_j$, such that the following holds:

$$\psi(z) - (T\psi)(z) = \frac{L}{z - V_q}, \tag{108}$$

where $L = \sum_{j=1}^{r} \beta_j \frac{\tilde{\partial}^V \rho_j}{\partial V_q}$. By (54),

$$\frac{\tilde{\partial} \rho_j}{\partial \sigma} = \frac{\tilde{\Gamma}_2^j}{\sigma}, \tag{109}$$

where $\tilde{\Gamma}_2^j$ and also $\tilde{\Gamma}_1^j$ are defined by the expansion $\tilde{B}_j(z) = \frac{\tilde{\Gamma}_1^j}{z} + \frac{\tilde{\Gamma}_2^j}{z^2} + O(\frac{1}{z^3})$ at infinity. Therefore, if M_1, M_2 are defined by $\psi(z) = \frac{M_1}{z} + \frac{M_2}{z^2} + O(\frac{1}{z^3})$ at infinity, then

$$M_2 = \sum_{j=1}^{r} \beta_j \frac{\tilde{\partial} \rho_j}{\partial \sigma} = 0. \tag{110}$$

Now, by (81),

$$T\psi(z) = \frac{M_1}{\sigma} \frac{1}{z} + (M_2 + \frac{mM_1}{\sigma}) \frac{1}{z^2} + O(\frac{1}{z^3}), \tag{111}$$

and

$$\psi(z) - T\psi(z) = \frac{\sigma - 1}{\sigma} M_1 \frac{1}{z} - \frac{mM_1}{\sigma} \frac{1}{z^2} + O(\frac{1}{z^3}). \tag{112}$$

But $m = 0$, which, together with (112) and (108) imply

$$L = M_1(1 - \frac{1}{\sigma}), \qquad LV_q = 0. \tag{113}$$

Since $V_q \neq 0$, then $L = 0$. In other words, ψ is a fixed point of T. It satisfies the conditions of Proposition 6. Indeed, since $\sigma \neq 1$, (113) then gives us that $M_1 = 0$, and then $\psi(z) = O(1/z^3)$. Applying Proposition 6 (where the assumptions (i)(a)-(ii) hold), we get a contradiction.

Remaining cases are quite similar.

(H_∞^{attr}). One can assume $k = q$ and assuming that the rank of the martix $\mathbf{O}^{\mathbf{attr}}$ is less than r and using the notations of the previous case, we obtain that $L = 0$, i.e., ψ is a non-trivial fixed point of T. Applying Proposition 6 with the assumptions (i)(b)-(ii), get a contradiction.

(NN_∞). One can assume $k = q$ and assume that the rank of the matrix $\mathbf{O}^{\mathbf{neutral}}$ is less than r. Using the notations from the first case, there exists a linear combination $\psi = \sum_{j=1}^{r} \beta_j \tilde{B}_j$, such that

$$\psi(z) - (T\psi)(z) = \frac{L}{z - V_q}, \tag{114}$$

where $L = \sum_{j=1}^r \beta_j \frac{\tilde\partial^V \rho_j}{\partial V_q}$. If M_1, M_2 are defined by $\psi(z) = \frac{M_1}{z} + \frac{M_2}{z^2} + O(\frac{1}{z^3})$ at infinity, then, by (81),

$$T\psi(z) = \frac{M_1}{z} + \frac{M_2 + mM_1}{z^2} + O(\frac{1}{z^3}), \tag{115}$$

and

$$\psi(z) - T\psi(z) = -\frac{mM_1}{z^2} + O(\frac{1}{z^3}). \tag{116}$$

This, along with (114), gives us $L = 0$. That is, ψ is a non-zero fixed point of T. In turn, it implies that $mM_1 = 0$. If $M_1 = 0$, then $\psi(z) = O(1/z^2)$, and if $M_1 \neq 0$, then $m = 0$. In either case, Proposition 6 applies. It gives a contradiction in this case, too.

(ND_∞), i.e. $\sigma = 1$ and $m = 0$. One can assume that $k = q-1$, $l = q$. Now, assuming that the rank of the matrix is less than r, we get that the vectors $(\frac{\tilde\partial^V \rho_j}{\partial V_1}, ..., \frac{\tilde\partial^V \rho_j}{\partial V_{q-2}})$, $1 \leq j \leq r$, are linearly dependent. In the previous notations, there exists a linear combination $\psi = \sum_{j=1}^r \beta_j \tilde B_j$, such that

$$\psi(z) - (T\psi)(z) = \frac{L_{q-1}}{z - V_{q-1}} + \frac{L_q}{z - V_q}, \tag{117}$$

where

$$L_i = \sum_{j=1}^r \beta_j \frac{\tilde\partial^V \rho_j}{\partial V_i}, \quad i = q-1, q.$$

If M_1, M_2 are defined by $\psi(z) = \frac{M_1}{z} + \frac{M_2}{z^2} + O(\frac{1}{z^3})$ at infinity, then, from (81) with $\sigma = 1$, $m = 0$,

$$\psi(z) - T\psi(z) = O(\frac{1}{z^3}). \tag{118}$$

This, along with (117), gives us two linear relations $L_{q-1} + L_q = 0$, $L_{q-1}V_{q-1} + L_q V_q = 0$. But $V_{q-1} \neq V_q$. Hence, $L_{q-1} = L_q = 0$. In other words, ψ is a non-zero fixed point of T. By Proposition 6, it is a contradiction again.

Comment 11 *This proof demonstrates also the inequalities from Comment 10. Indeed, otherwise the rows of* $\mathbf{O}$ *are again linearly dependent, and the proof above applies. Observe however, that for this purpose already the formal identity (22) of Theorem 3 (i.e., with some coefficients* L_j *without their connections to parameter spaces) would be sufficient. This approach is somewhat similar to the one developed first in [9] for the proof of the Fatou-Shishikura inequality, where also more degenerated cases of the inequality are covered.*

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